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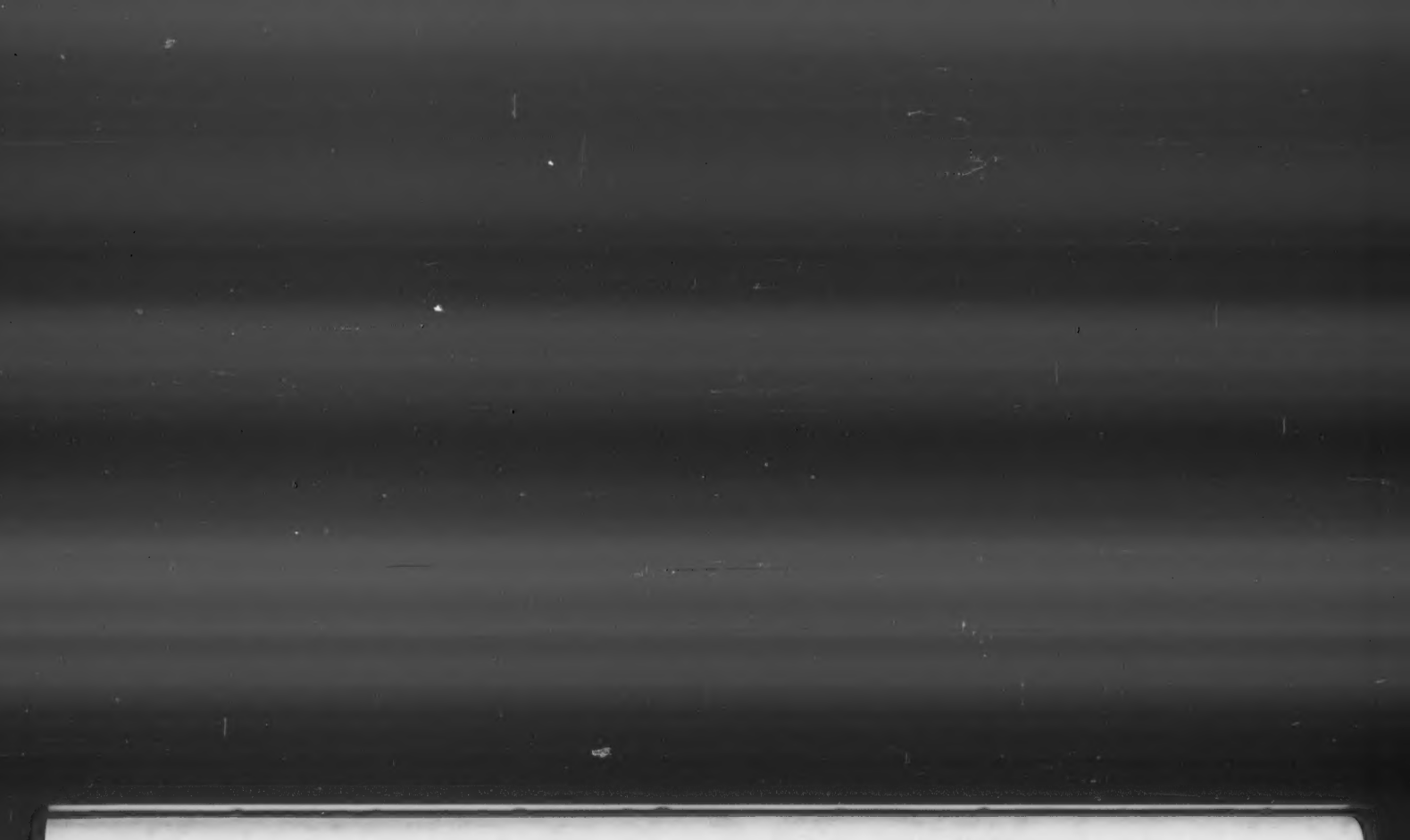

CLOSED INTEGRAL CURVES IN 3-SPACE AND ISOTOPIC TWO-DIMENSIONAL DEFORMATIONS

HERBERT SEIFERT

If there is given an isotopic deformation of a two-dimensional euclidean plane E onto itself, one can adjoin to every point P of E a certain closed curve, the so-called "indicatrix" of P (§1). If the indicatrix does not pass through P, we introduce the order of P relative to its indicatrix as the "rotation number" of the deformation at P. A relation between the rotation numbers in different points of E (§3) and a formula for the rotation number in the general case of a "bounded deformation" (§4) is established. This formula admits an application to the problem of closed integral curves of continuous vector fields in the 3-dimensional sphere.

1. **Indicatrix and rotation number.** An *isotopic deformation* of the plane E is a set $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 1)$ of topological mappings of E onto itself such that A_0 is the identical mapping and that $A_\lambda(P)$ depends continuously on λ and P. The curve $A_\lambda(P)$ $(0 \leqq \lambda \leqq 1)$ is called the trajectory of P. It is unessential that we confine λ to the interval $0 \leqq \lambda \leqq 1$. Sometimes it will be advisable to take another closed interval $a \leqq \lambda \leqq b$.

Given the deformation $\{A_\lambda\}$ we can adjoin to every point P of the plane E an indicatrix, namely the curve

$$A_\lambda A_1 A_\lambda^{-1}(P) \qquad (0 \leqq \lambda \leqq 1). \tag{1}$$

This curve is closed since (1) reduces to $A_1(P)$ both for $\lambda = 0$ and $\lambda = 1$. It may happen (for example, when the mappings A_λ are commutative) that the indicatrix consists only of the point $A_1(P)$.

The point P lies on its indicatrix if and only if for a suitable λ

$$A_\lambda A_1 A_\lambda^{-1}(P) = P$$

or

$$A_1 A_\lambda^{-1}(P) = A_\lambda^{-1}(P). \tag{2}$$

Putting

$$A_\lambda^{-1}(P) = Q, \tag{3}$$

we see from (2) that Q is a fixed point of A_1 and from (3) that P lies on the trajectory of Q. Thus the two statements "P lies on its indi-

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catrix" and "P lies on the trajectory of a fixed point of A_1" are equivalent.

If P lies on its indicatrix, we call P an *exceptional point*. Let N be the set of all exceptional points. N is empty if and only if A_1 is without fixed points. N is closed. For if P is a point of $E-N$, P is not on its indicatrix. Then the same is true for all points of a sufficiently small neighborhood of P, as the indicatrix varies continuously with P. Thus all points of a sufficiently small neighborhood of P belong to $E-N$, that is N is closed.

Let P be not an exceptional point. Then P has a certain order relative to its indicatrix. The order of a point P relative to a closed curve q which does not pass through P is defined to be $1/2\pi$ times the total change of the angle between the vector[1] $\langle PQ\rangle$ and a fixed direction when Q traverses the curve q. Of course the order depends on the orientation of the plane. We always consider a triangle OAB positively oriented with O being the origin, A a point on the positive x-axis, B a point on the positive y-axis. The order of P relative to its indicatrix will be called the *rotation number* $\omega(P)$ of the deformation $\{A_\lambda\}$ at the point P. The plane E is decomposed by the set N of all exceptional points into a number of regions (components of $E-N$). If P is allowed to vary continuously in a component of $E-N$, then the indicatrix of P changes continuously and never passes through P. Hence the rotation number is constant on every component of $E-N$. In the special case that N is the empty set or, which is the same, the mapping A_1 has no fixed points, the rotation number is constant throughout the plane.

EXAMPLES:

I.

$$A_\lambda: \quad x' = x + \lambda, \quad y' = y.$$

A_1 has no fixed points. The indicatrix of the point (x, y) shrinks to the point $(x+1, y)$. The rotation number is equal to zero everywhere.

II.

$$(4) \quad A_\lambda: \quad \begin{aligned} x' &= x\cos 2k\pi\lambda - y\sin 2k\pi\lambda + \lambda, \\ y' &= x\sin 2k\pi\lambda + y\cos 2k\pi\lambda \end{aligned} \qquad (k \text{ an integer}).$$

A_1 is the translation $x'=x+1$, $y'=y$. Therefore

$$A_\lambda A_1 A_\lambda^{-1}: \quad x' = x+\cos 2k\pi\lambda, \quad y' = y + \sin 2k\pi\lambda,$$

the transform of a translation A_1 by a rotation A_λ being again a trans-

[1] The vector from the initial point P to the end point Q will be designated by $\langle PQ\rangle$, the usual notation, an arrow over the points PQ, being difficult to print.

lation. Thus the indicatrix of a point P is the unit circle with center P, k times traversed. The rotation number is equal to k throughout the plane.

III.

(5) A_λ: $\begin{aligned} x' &= x\cos 2k\pi\lambda - y\sin 2k\pi\lambda, \\ y' &= x\sin 2k\pi\lambda + y\cos 2k\pi\lambda \end{aligned}$ (k not an integer).

A_1 is the rotation around the origin through the angle $2k\pi$. The origin is the only exceptional point. In all points the rotation number is defined and equal to zero because of the commutativity of the rotations (5).

2. **Three-dimensional interpretation of the rotation numbers.** We consider the plane E in which the deformation $\{A_\lambda\}$ is given as the plane $z=0$ of an orthogonal xyz-coordinate system. Let B_λ be the translation

(6) $$B_\lambda: \quad x' = x,\ y' = y,\ z' = z + \lambda.$$

Then $B_\lambda A_\lambda$ maps the plane $z=0$ onto the plane $z=\lambda$. For every $P\in E$ the points

(7) $$B_\lambda A_\lambda(P) = Q_\lambda \qquad (0 \leqq \lambda \leqq 1)$$

form a curve which lies in the three-dimensional set Z

(8) $$Z: \quad 0 \leqq z \leqq 1, \qquad -\infty < x < +\infty, \ -\infty < y < +\infty$$

and whose parameter λ coincides with the z-coordinate. The orthogonal projection of the curve (7) into the plane E is identical with the trajectory of P. (7) will be called the *streamline* of the point P. As the mapping $B_\lambda A_\lambda$ of the plane $z=0$ onto the plane $z=\lambda$ is one-to-one, there is exactly one streamline through every point of Z.

If one identifies every two points $(x, y, 0)$ and $(x, y, 1)$, one can continue on a streamline arbitrarily far: when one has arrived in a point $(x, y, 1)$, one continues on the streamline going out from $(x, y, 0)$. In particular one can start in an arbitrary point Q_λ with z-coordinate λ and traverse the streamline until z has an increment of $\Delta z=1$. One arrives thus in a point Q_λ' in the plane $z=\lambda$. The mapping $Q_\lambda \to Q_\lambda'$ of Z onto itself will be called the *return mapping* T of the deformation $\{A_\lambda\}$, the vector $\langle Q_\lambda Q_\lambda' \rangle$ which is attached to the point Q_λ is termed the *return vector* of the point Q_λ. T is obviously characterized by the following properties:

(a) T induces the mapping A_1 in the plane $z=0$.

(b) T carries streamlines into streamlines.

(c) T maps every plane $z=\lambda$ onto itself.

In the plane $z=\lambda$ the mapping T induces the mapping $A_\lambda A_1 A_\lambda^{-1}$ provided that one pays attention to the transformation of the coordinates x, y only. To prove this we introduce the points P and P' in which the streamlines through Q_λ and Q_λ' intersect the plane $z=0$. From (7) we have

$$Q_\lambda = B_\lambda A_\lambda(P); \qquad Q_\lambda' = B_\lambda A_\lambda(P')$$

and from property (a)

$$P' = A_1(P).$$

It follows that

$$Q_\lambda' = B_\lambda \cdot A_\lambda A_1 A_\lambda^{-1} \cdot B_\lambda^{-1}(Q_\lambda), \tag{9}$$

which was to be proved.

In particular T induces in the plane $z=1$ (as well as in the plane $z=0$) the mapping A_1. Because of the properties (b) and (c), a point Q of Z remains fixed under the return mapping T if and only if all points of the streamline through Q remain fixed. So the fixed points of T are the points of the streamlines passing through the fixed points of the mapping A_1 of the plane $z=0$. These streamlines will be termed the *fixed lines* of T. By orthogonal projection of the fixed lines onto $z=0$ one obtains exactly the set N of exceptional points (§1). Now we can give a 3-dimensional interpretation of the indicatrix and the rotation number $\omega(P)$ of a point P in the plane $z=0$. *Let p be the segment $0 \leqq z \leqq 1$ through P parallel to the z-axis, $T(p)$ the image of p under the return mapping T, then the orthogonal projection of $T(p)$ onto the plane $z=0$ is the indicatrix of P.*

This is an immediate consequence of the definition of the indicatrix and of the fact that T induces the mapping $A_\lambda A_1 A_\lambda^{-1}$ in the plane $z=\lambda$. *So the rotation number $\omega(P)$ is up to the factor 2π equal to the total change of the return vector when its initial point traverses the segment p.*

3. **Relation between the rotation numbers in different points.** Henceforth we shall assume that the mapping A_1 which is defined in the plane $z=0$ only has isolated fixed points F_ν $(\nu=1, 2, \cdots)$. Let $\tilde{f}_\nu$ be the fixed line with the initial point F_ν oriented in the direction of increasing values of z. The end point of $\tilde{f}_\nu$ is the point of the plane $z=1$ which has the same xy-coordinates as F_ν.

Every point P of the three-dimensional domain $Z^*=Z-\bigcup \tilde{f}_\nu$ is the initial point of a return vector $\langle PQ \rangle$ which is different from zero. When P traverses a singular 1-simplex $s \subset Z^*$ there is a certain total

change of the angle between $\langle PQ\rangle$ and the positive x-axis. This change, divided by 2π, is called the *rotation number* $\omega(s)$ of the 1-simplex s. More generally we can adjoin to every 1-chain $\sigma=s_1+\cdots+s_k\in Z^*$ a rotation number, namely $\omega(\sigma)=\omega(s_1)+\cdots+\omega(s_k)$. $\omega(\sigma)$ is an integer if σ is a cycle. As the return vector $\langle PQ\rangle$ depends continuously on P, there is for every $P\subset Z^*$ a neighborhood U_P such that $\omega(\sigma)=0$ for every cycle in U_P. It follows that the rotation number is zero for every cycle in Z^* which is homologous to zero in Z^*.

Let k_ν be a circle in the plane $z=0$ with center F_ν such that F_ν is the only fixed point of A_1 on the closed interior of k_ν. Choose the orientation of k_ν so that the order of F_ν relative to k_ν is $+1$. As the return vector in the plane $z=0$ runs from P to $A_1(P)$ we find that $\omega(k_\nu)$ is the fixed point index γ_ν of F_ν under the mapping A_1:

$$\omega(k_\nu)=\gamma_\nu. \tag{10}$$

The circles k_ν form a homology basis of dimension 1 of Z^*. For Z^* is the topological product of $E-\bigcup F_\nu$ and the closed unit segment. Thus for an arbitrary 1-cycle $c\subset Z^*$ there is a homology

$$c\sim\sum\chi_\nu k_\nu \quad \text{on} \quad Z^*, \tag{11}$$

where of course only a finite number of coefficients χ_ν are not 0. χ_ν is obviously the intersection number of $\bar{f}_\nu$ and a 2-chain $b\subset Z$ with boundary c:

$$\chi_\nu=S(\bar{f}_\nu, b), \tag{12}$$

$$\dot{b}=c. \tag{13}$$

The right side of (12) has a meaning provided that b intersects neither the initial points nor the end points of the paths $\bar{f}_\nu$. This always can be assumed. In formula (12) we make use of the orientation (O, O_1, O_2, O_3) of Z where O, O_1, O_2, O_3 are respectively the origin and points on the positive x, y, z-axis. With this orientation the intersection number of the oriented z-axis and the oriented xy-plane is $+1$.

For the rotation number $\omega(c)$ we have from (10) and (11)

$$\omega(c)=\sum\chi_\nu\gamma_\nu. \tag{14}$$

Let f_ν be the orthogonal projection of $\bar{f}_\nu$ into the plane $E\,(z=0)$, and U, V two points of $E-\bigcup f_\nu$ whose rotation numbers we wish to compare. If u and v are the unit segments through U and V parallel to the z-axis, we have from the result of §2

$$\omega(U)=\omega(u), \qquad \omega(V)=\omega(v). \tag{15}$$

We draw from U to V in the plane $z=0$ a path q which avoids the other fixed points. By the translation $x'=x$, $y'=y$, $z'=z+1$, q is carried into the path q_1 in the plane $z=1$. Then

$$c = q + v - q_1 - u$$

is a 1-cycle in Z^*. Hence

$$\omega(c) = \omega(v) - \omega(u) \tag{16}$$

as the return vectors in corresponding points of q and q_1 are equal. Let b be the 2-chain that is described by q under the translation into q_1. By the projection of Z onto E the intersection points of b and $\tilde{f}_\nu$ are carried into the intersection points of q and f_ν. According to our choice of the orientations of E and Z the indices of the intersection points are not altered by the projection. So we have from (12)

$$\chi_\nu = S(\tilde{f}_\nu, b) = S(f_\nu, q). \tag{17}$$

From (14), (15), (16), (17) we deduce the following theorem.

THEOREM 1. *If U and V are not exceptional points relative to the deformation $\{A_\lambda\}$ and if the fixed points F_ν of A_1 have no limit point, then the rotation numbers $\omega(U)$ and $\omega(V)$ are connected by the relation*

$$\omega(V) - \omega(U) = \sum_1^\infty \gamma_\nu S(f_\nu, q), \tag{18}$$

in which γ_ν is the fixed point index of F_ν under the mapping A_1, f_ν the trajectory of F_ν under the deformation $\{A_\lambda\}$ and $S(f_\nu, q)$ the intersection number of f_ν and a curve q running from U to V.

In the formulation of the theorem the condition that q should avoid the fixed points F_ν is obviously unnecessary.

4. **Bounded deformations.** An isotopic deformation $\{A_\lambda\}$ of the plane is called *bounded* if there is an $\epsilon>0$ such that the diameter of the trajectory of every point is less than ϵ. More precisely $\{A_\lambda\}$ is termed an ϵ-deformation.

We shall generalize this concept in so far as we shall only assume that the deformation $\{A_\lambda\}$ $(0\leqq\lambda\leqq 1)$ is defined in a certain domain G of the plane E (not necessarily in the whole plane). In other words we shall assume that (a) $A_\lambda(0\leqq\lambda\leqq 1)$ is a topological mapping of G into the plane E (not necessarily onto G); (b) A_λ depends continuously on P and λ; (c) A_0 is the identity mapping; (d) for every point $P\in G$ the diameter of the trajectory of P under the deformation is less than ϵ. Then $\{A_\lambda\}$ is called an isotopic ϵ-deformation of G. If for

some $P \in G$ the curve $A_\lambda A_1 A_\lambda^{-1}(P)$ $(0 \leqq \lambda \leqq 1)$ exists, it is called the indicatrix of P. If P does not lie on its indicatrix, the order of P relative to its indicatrix is called the rotation number of the deformation at P. These definitions obviously agree with those of §1.

Let in particular G be the square

$$|x - x_0| < a, \qquad |y - y_0| < a \qquad (a > 0) \tag{19}$$

and

$$\epsilon < a/2.$$

Then the squares

$$|x - x_0| < a - \epsilon, \qquad |y - y_0| < a - \epsilon, \tag{20}$$

$$|x - x_0| < a - 2\epsilon, \qquad |y - y_0| < a - 2\epsilon \tag{21}$$

exist. A_λ maps (19) onto a point set which contains (20). So A_λ^{-1} is defined on (20). Similarly, $A_\lambda A_1 A_\lambda^{-1}$ is defined on (21). For A_λ^{-1} maps (21) into (20); A_1 maps (20) into (19), and A_λ is defined everywhere on (19).

LEMMA 1. *If* $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 1)$ *is an* ϵ*-deformation of the square* (19) *such that*

$$\epsilon < a/3, \tag{22}$$

and for every P *of* (19)

$$x(P) < x(A_1(P)), \tag{23}$$

$x(P)$ *being the abscissa of* P, *then in all points of the square* (21) *the rotation number is defined and equal to zero.*

PROOF. We have shown that for every point P of (19) the indicatrix $A_\lambda A_1 A_\lambda^{-1}(P)$ is defined. It does not pass through P because otherwise for a suitable λ, $A_\lambda^{-1}(P)$ would be a fixed point of A_1, contrary to (23). Hence in all points of (21) the rotation number is defined and consequently constant. So it will be sufficient to calculate it for a single point, say for

$$Q = (x_0 - \epsilon, 0).$$

Let

$$\overline{Q} = (x_0 + \epsilon, 0) \quad \text{and} \quad Q' = A_1(Q).$$

From (23) and the fact that Q and Q' have a distance less than ϵ, it follows that

(24) $$x(Q) < x(Q') < 0.$$

Let R_μ be the point that divides the segment $Q'\overline{Q}$ in the ratio μ: $(1-\mu)$ $(0 \leqq \mu \leqq 1)$. We consider the vector

(25) $$v(\lambda, \mu) = \begin{cases} \langle A_\lambda(Q) A_\lambda(R_\mu) \rangle & \text{for } 0 \leqq \lambda \leqq 1,\ 0 \leqq \mu \leqq 1, \\ \langle A_{\lambda(1+\mu)}(Q) A_\lambda A_1 A_\lambda^{-1} A_{\lambda(1+\mu)}(Q) \rangle & \\ & \text{for } 0 \leqq \lambda \leqq 1,\ -1 \leqq \mu \leqq 0. \end{cases}$$

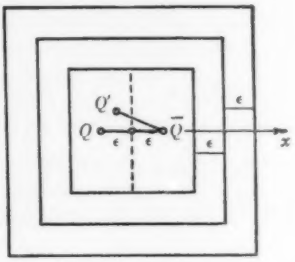

FIG. 1

The two definitions coincide for $\mu=0$ since $R_0=Q'=A_1(Q)$ and $A_{\lambda(1+0)}=A_\lambda$. $v(\lambda, \mu)$ vanishes nowhere in the rectangle

(26) $$0 \leqq \lambda \leqq 1, \qquad -1 \leqq \mu \leqq 1.$$

In fact, from (24) and

(27) $$x(Q') \leqq x(R_\mu),$$

it follows that Q and R_μ are different. So the same is true for $A_\lambda(Q)$ and $A_\lambda(R_\mu)$, that is, $v(\lambda, \mu) \neq 0$ for $0 \leqq \lambda \leqq 1$, $0 \leqq \mu \leqq 1$. Similarly, for $-1 \leqq \mu \leqq 0$ the point $S=A_\lambda^{-1}A_{\lambda(1+\mu)}(Q)$ lies within the circle of Q with radius 2ϵ, A_λ^{-1} and $A_{\lambda(1+\mu)}$ being ϵ-mappings. From (22) and (23) we deduce that $A_1(S)$ is contained in (19) and

$$S \neq A_1(S).$$

It follows by multiplication with A_λ

$$A_\lambda(S) \neq A_\lambda A_1(S),$$

that is, $v(\lambda, \mu) \neq 0$ for $0 \leqq \lambda \leqq 1$; $-1 \leqq \mu \leqq 0$. Next one can verify that on the edges $\lambda=0$, $\lambda=1$, $\mu=1$ of the rectangle (26) the end point of the vector $v(\lambda, \mu)$ always has greater abscissa than the initial point. Indeed we have for $\lambda=0$

$$v(0, \mu) = \begin{cases} \langle QR_\mu \rangle & (0 \leqq \mu \leqq 1), \\ \langle QQ' \rangle & (-1 \leqq \mu \leqq 0) \end{cases}$$

and

$$x(Q) < x(Q') \leqq x(R_\mu)$$

from (24) and (27). Similarly for $\lambda = 1$

$$v(1, \mu) = \begin{cases} \langle Q'A_1(R_\mu) \rangle & (0 \leqq \mu \leqq 1), \\ \langle A_{1+\mu}(Q)A_1A_{1+\mu}(Q) \rangle & (-1 \leqq \mu \leqq 0) \end{cases}$$

and

$$x(Q') \leqq x(R_\mu) < x(A_1(R_\mu)),$$
$$x(A_{1+\mu}(Q)) < x(A_1A_{1+\mu}(Q))$$

from (27) and (23). Finally for $\mu = 1$

$$v(\lambda, 1) = \langle A_\lambda(Q)A_\lambda(\overline{Q}) \rangle$$

and

$$x(A_\lambda(Q)) < 0, \qquad x(A_\lambda(\overline{Q})) > 0$$

since $x(Q) = x_0 - \epsilon$, $x(\overline{Q}) = x_0 + \epsilon$ and A_λ is an ϵ-mapping.

Now we consider $v(\lambda, \mu)$ for $\mu = -1$. We have

$$v(\lambda, -1) = \langle QA_\lambda A_1 A_\lambda^{-1}(Q) \rangle.$$

This is the vector that leads from Q to the point (λ) on the indicatrix of Q. So the rotation number $\omega(Q)$ is given (up to a factor 2π) by the total change of the angle between the positive x-axis and the vector $v(\lambda, -1)$ when λ goes from 0 to 1. Since $v(\lambda, \mu)$ does not vanish in the rectangle (26) we come to the same result when we continue $v(\lambda, \mu)$ along the three other edges of the rectangle, but here the end point of $v(\lambda, \mu)$ has always greater abscissa than the initial point. Therefore

$$\omega(Q) = 0, \qquad \text{q.e.d.}$$

THEOREM 2. *Let $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 1)$ be a bounded isotopic deformation of the plane E, such that A_1 has at most isolated fixed points $F_1, F_2, \cdots$. If f_ν is the trajectory of F_ν, then for a point V which is on none of the f_ν the rotation number $\omega(V)$ is given by*

$$\omega(V) = \sum_1^\infty \gamma_\nu \theta_\nu, \tag{28}$$

where θ_1 is the order of V relative to f, and γ_1 is the fixed point index of F, at the mapping A_1.

An immediate consequence of Theorem 2 is the following:

THEOREM 3. *If $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 1)$ is a bounded isotopic deformation of the plane and A_1 is without fixed points, then the rotation number of the deformation is zero in every point of the plane.*

PROOF OF THEOREM 2. Because of the boundedness of the deformation there is a $\delta > 0$ so that for every point the diameter of the trajectory is less than δ. We choose the coordinate system so that V gets the coordinates $x = -4\delta$, $y = 0$.

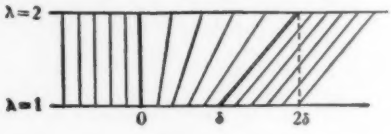

FIG. 2

We extend the deformation $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 1)$ to a deformation whose parameter λ runs from 0 to 3 by the following definition: For $1 \leqq \lambda \leqq 2$ let $\{A_\lambda A_1^{-1}\}$ be the δ-deformation

$$
\begin{aligned}
&x' = x, && y' = y && (x \leqq 0),\\
&x' = \lambda x, && y' = y && (0 \leqq x \leqq \delta),\\
&x' = x + (\lambda - 1)\delta, && y' = y && (x \geqq \delta)
\end{aligned}
\tag{29}
$$

whose streamlines are shown by Fig. 2.

For $2 \leqq \lambda \leqq 3$ let $\{A_\lambda A_2^{-1}\}$ be an isotopic η-deformation $(\eta < \delta)$ and A_2 a (necessarily topological) simplicial mapping of the plane with only isolated fixed points.[2] In the following we shall restrict η by some further conditions.

The deformation $\{A_\lambda\}$ $(0 \leqq \lambda \leqq 3)$ is denoted by $A_{(0,3)}$ and the meaning of $A_{(0,1)}$ and $A_{(0,2)}$ is similar. $A_{(0,1)}$, $A_{(0,2)}$, $A_{(0,3)}$ is respectively a δ-, 2δ-, 3δ-deformation. All quantities (fixpoints, rotation numbers, trajectories) concerning $A_{(0,2)}(A_{(0,3)})$ will be distinguished from the corresponding quantities of $A_{(0,1)}$ by a prime (two

[2] Every typological mapping A_2 of the plane onto itself can be carried into a simplicial mapping by an isotopic η-deformation (η arbitrarily small). (See Graeub, *Semilineare Abbildungen*, to appear in Sitzungsberichte der Heidelberger Akademie der Wissenschaften. This deformation may be omitted if A_2 has only isolated fixed points.

primes). For example, the fixed points of A_2 will be designated by F_ν'', and f_ν'' is the trajectory of F_ν'' under $A_{(0,3)}$.

We consider these deformations

(a) in the left half-plane L: $x<0$,

(b) in the right half-plane R: $x>2\delta$,

(c) in the middle area M: $0\leqq x\leqq 2\delta$.

The fixed points F_ν of A_1 are denoted by F_{l_ν}, F_{r_ν}, or F_{m_ν}, according as they belong to L, R, or M. The fixed points F_ν' of A_2 and F_ν'' of A_3 will be classified and denoted in a similar way.

If P is a point of R, we have $x(P)-x(A_1(P))<\delta$ since $A_{(0,1)}$ is a δ-deformation; moreover $x(A_2(P))-x(A_1(P))=\delta$ because of (29), therefore

$$x(P) < x(A_2(P)). \tag{30}$$

It follows that for all points P of a bounded closed subset of R and for sufficiently small η the inequality

$$x(P) < x(A_3(P)) \tag{31}$$

holds, for example for the points of the square

$$|x - 13\delta| \leqq 10\delta; \qquad |y| \leqq 10\delta. \tag{32}$$

Since $A_{(0,3)}$ is a 3δ-deformation, we have for the rotation number $\omega''(U)$ at the center $U=(13\delta, 0)$ of the square (32) because of Lemma 1

$$\omega''(U) = 0. \tag{33}$$

At the point $V=(-4\delta, 0)$ the rotation number $\omega(V)$ relative to the deformation $A_{(0,1)}$ is defined by hypothesis, that is, the indicatrix $A_\lambda A_1 A_\lambda^{-1}(V)$ $(0\leqq\lambda\leqq 1)$ does not pass through V. So the same is true for the indicatrix $A_\lambda A_2 A_\lambda^{-1}(V)$ $(0\leqq\lambda\leqq 3)$ relative to the deformation $A_{(0,3)}$, provided that η is sufficiently small. For under this condition the two indicatrices will differ arbitrarily little. So $\omega''(V)$ is defined and equal to $\omega(V)$:

$$\omega''(V) = \omega(V). \tag{34}$$

Now we can apply Theorem 1 on the deformation $A_{(0,3)}$ and the points U, V:

$$\omega(V) = \omega''(V) = \omega''(U) + \sum \gamma_\nu'' S(f_\nu'', q). \tag{35}$$

q may be chosen to be the straight segment UV. (33) and (35) imply

$$\omega(V) = \sum \gamma_{l_\nu}'' S(f_{l_\nu}'', q) + \sum \gamma_{m_\nu}'' S(f_{m_\nu}'', q) + \sum \gamma_{r_\nu}'' S(f_{r_\nu}'', q). \tag{36}$$

Since $A_{(0,3)}$ is a 3δ-deformation and there are no fixed points of A_3 in the square (32) because of (31), we have $S(f''_{r,\nu}, q)=0$ so that the third term in (36) vanishes. But also the second term on the right side of (36) is zero. For $f''_{m,\nu}$, being a trajectory of a point of M, lies in the strip $-3\delta<x<5\delta$ and consequently has with q the intersection number 0. Since the trajectory $f''_{i,\nu}$ is in the half-plane $x<3\delta$, the intersection number $S(f''_{i,\nu}, q)$ is not altered when we replace q by the ray drawn from V in the direction of the positive x-axis. In other words, $S(f''_{i,\nu}, q)$ is equal to the order $\theta''_{i,\nu}$ of V relative to $f''_{i,\nu}$. So (36) implies

$$\omega(V) = \sum \gamma''_{i,\nu}\theta''_{i,\nu}. \tag{37}$$

On the other hand we consider expression (28): $\sum\gamma_\nu\theta_\nu$. Let K be the closed circular disk with center V whose radius ρ is chosen so that $3\delta<\rho<4\delta$ and that no fixed point of A_1 is on the boundary of K. If F_ν is a fixed point of A_1 which does not lie on K, the trajectory f_ν of F_ν cannot encircle V, since the diameter of f_ν is less than δ. So in (28) we have only to consider the fixed points on K. Let these be $F_1, F_2, \cdots, F_p$. We choose $\beta>0$ so small that the open β-neighborhoods $U_\beta(F_\nu)$ for $\nu=1, 2, \cdots, p$ belong to K and do not intersect one another. Since A_1 has no fixed point on the closed set

$$K - \bigcup_{\nu=1}^{p} U_\beta(F_\nu) = H,$$

there is an $\alpha>0$ such that for every $P\in H$ the distance of P and $A_1(P)$ is not less than α. If we choose $\eta<\alpha$, then also A_3 has no fixed points on H and we have

$$\sum_{i=1}^{h_\nu} \gamma''_{\nu,i} = \gamma_\nu, \tag{38}$$

if $F''_{\nu,1}, F''_{\nu,2}, \cdots, F''_{\nu,h_\nu}$ are the fixed points of A_3 on $U_\beta(F_\nu)$ and $\gamma''_{\nu,1}, \gamma''_{\nu,2}, \cdots, \gamma''_{\nu,h_\nu}$ are their indices. Moreover, the trajectory of $F''_{\nu,i}$ under the deformation $A_{(0,3)}$ differs arbitrarily little from the path of F_i under $A_{(0,1)}$ (provided that η is sufficiently small). Therefore we may assume that the orders $\theta''_{\nu,i}$ and θ_ν of V relative to these trajectories are equal:

$$\theta''_{\nu,i} = \theta_\nu. \tag{39}$$

So we obtain

$$\sum_{\nu=1}^{\infty} \gamma_\nu\theta_\nu = \sum_{\nu=1}^{p} \gamma_\nu\theta_\nu = \sum_{\nu=1}^{p}\sum_{i=1}^{h_\nu} \gamma''_{\nu,i}\theta''_{\nu,i}. \tag{40}$$

The right sides of (37) and (40) differ only by terms $\gamma_\nu''\vartheta_\nu''$ that correspond to fixed points F_ν'' not on K. But these have a distance $>\rho>3\delta$ from V, such that the corresponding trajectories f_ν'', being less than 3δ in diameter, cannot encircle V. So we have $\vartheta_\nu''=0$ for these F_ν''. It follows that the right sides of (37) and (40) are equal, hence

$$\omega(V) = \sum_{\nu=0}^{\infty} \gamma_\nu \vartheta_\nu,$$

q.e.d.

Theorems 1 and 2 admit generalizations for the case where the deformations $\{A_\lambda\}$ $(0\leqq\lambda\leqq 1)$ are only defined in a certain open region G, not necessarily in the whole plane.

THEOREM 1′. *Let U and V be two points of a plane open region G. Then there exists an $\epsilon>0$ with the following property: Formula* (18) *holds for every isotopic ϵ-deformation $\{A_\lambda\}$ $(0\leqq\lambda\leqq 1)$ of G for which A_1 has at most isolated fixed points (provided that the rotation numbers $\omega(U)$ and $\omega(V)$ are defined). $q\subset G$ is a path leading from U to V.*

THEOREM 2′. *Let V be a point of a plane open region G. Then there is an $\epsilon>0$ with the following property: Formula* (28) *holds for every isotopic ϵ-deformation $\{A_\lambda\}$ $(0\leqq\lambda\leqq 1)$ of G for which A_1 has at most isolated fixed points (provided that the rotation number $\omega(V)$ is defined).*

THEOREM 3′. *Let V be a point of a plane open region G. Then there is an $\epsilon>0$, such that $\omega(V)=0$ for every isotopic ϵ-deformation of G for which A_1 is without fixed points.*

Theorem 3′ is contained in Theorem 2′. The proofs of Theorems 1′ and 2′ differ only unessentially from those of Theorems 1 and 2. One only has to choose ϵ so small that V respectively U and V are sufficiently far from the boundary of G (compared with ϵ) in order that the constructions made in the proofs of Theorems 1 and 2 become possible. In the proof of Theorem 2′ use has to be made of the fact that every homeomorphic mapping of G into the plane can be deformed by an isotopic η-deformation (η arbitrarily small) into a simplicial mapping with only isolated fixed points.

5. Closed integral curves of vector fields in the three-dimensional sphere. If at every point of a differentiable n-dimensional manifold there is given a nonvanishing tangent vector which varies continuously with that point we speak of a continuous vector field. The necessary and sufficient condition for the existence of a continuous

vector field in a closed manifold is the vanishing of the Euler characteristic. A differentiable curve which at each of its points is tangent to the field vector at that point is called an integral curve of the field. It is unknown if every continuous vector field of the three-dimensional sphere S^3 contains a closed integral curve. In the following we consider on S^3 a special class of vector fields for which the existence of a closed integral curve is a consequence of Theorem 3'.

Let S^3 be the three-dimensional unit-sphere of the four-dimensional euclidean space

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{41}$$

We consider on S^3 the field C of the "Clifford-parallel" vectors

$$dx_1 = -x_2, \quad dx_2 = x_1, \quad dx_3 = -x_4, \quad dx_4 = x_3, \tag{42}$$

whose integral curves are great circles. They will be called the *Clifford-circles* of S^3. We choose the orientation of S^3 so that two different Clifford-circles (each of which has a definite orientation by (42)) have the looping coefficient $+1$.

Let $\tilde{C}$ be a continuous vector field on S^3 which differs sufficiently little from C, that is, the angle between the vector of C and that of $\tilde{C}$ is at every point of S^3 smaller than α (α sufficiently small). Moreover, we suppose that through every point of S^3 there passes exactly one integral curve. We shall prove that for sufficiently small α there exists at least one closed integral curve in $\tilde{C}$.

The integral curves of $\tilde{C}$ will be called the *streamlines* of S^3. Let β be a fixed number between 0 and $\pi/2$, P a point on S^3, c the Clifford circle through P, $U_\beta(c)$ the (open) β-neighborhood of c. We introduce in $U_\beta(c)$ coordinates x,y,z: P is the origin of the coordinate system, $x=$const., $y=$const. are the Clifford-circles in $U_\beta(c)$, $z=$const. are the two-dimensional great spheres orthogonal to c; x and y are Riemann normal coordinates on $z=0$ and z is the arclength on c, divided by 2π. There is a one-to-one correspondence between the points of $U_\beta(c)$ and the points of the set

$$x^2 + y^2 < \beta^2, \qquad 0 \leqq z < 1,$$

of euclidean xyz-space. The streamline which passes through P admits the representation

$$x = \phi(z), \qquad y = \psi(z).$$

$\phi(z)$, $\psi(z)$ are continuous functions of z and defined for $0 \leqq z \leqq 1$ if α was chosen sufficiently small. Let $D(P)$ be the point $x=\phi(1)$, $y=\psi(1)$, $z=1$ of S^3. Since P was an arbitrary point the representation

$P \to D(P)$ is defined everywhere on S^3. It can be considered as the final result of an isotopic deformation along the streamlines and therefore is a homeomorphic mapping with degree $+1$. We have to prove that D has fixed points.

For this purpose we choose once for all a Clifford-circle c^* and consider besides D a mapping T which is defined only in the $\beta/2$-neighborhood $U_{\beta/2}(c^*)$ of c^*. As previously the points of $U_\beta(c^*)$ are referred to coordinates x, y, z. Let $Q(x_0, y_0, z_0)$ be a point of $U_{\beta/2}(c^*)$ and

$$x = f(x_0, y_0, z_0, \lambda),\ y = g(x_0, y_0, z_0, \lambda),\ z = z_0 + \lambda \qquad (0 \leqq \lambda \leqq 1,\ x^2 + y^2 < \beta^2) \tag{43}$$

the streamline passing through $Q(x_0, y_0, z_0)$. For the existence of the representation (43) we must again suppose α to be small enough.

Let $T(Q)$ be the point

$$x = f(x_0, y_0, z_0, 1),\ y = g(x_0, y_0, z_0, 1),\ z = z_0 + 1.$$

In general the mappings $Q \to T(Q)$ and $Q \to D(Q)$ do not coincide on $U_{\beta/2}(c^*)$, for $D(Q)$ is on the great sphere which passes through Q and which is orthogonal to the Clifford-circle through Q, while $T(Q)$ is on the great sphere which passes through Q and which is orthogonal to the Clifford-circle c^*. However, if Q is on c^*, we have $D(Q) = T(Q)$ and consequently

$$T(c^*) = D(c^*). \tag{44}$$

(43) defines for $z_0 = 0$ an isotopic deformation

$$x = f(x_0, y_0, 0, \lambda), \qquad y = g(x_0, y_0, 0, \lambda) \qquad (0 \leqq \lambda \leqq 1) \tag{45}$$

of the two-dimensional region $x_0^2 + y_0^2 < \beta^2/4$.

In order to prove our theorem indirectly we shall suppose that there is no closed streamline. Then the final mapping ($\lambda = 1$) of (45) is without fixed points so that for sufficiently small α we may apply Theorem 3′. This means that the point ($x_0 = 0$, $y_0 = 0$) has the order 0 relative to its indicatrix. Now, the indicatrix is the projection of $T(c^*)$ along the Clifford-circles, and since every two Clifford-circles have the looping coefficients $+1$, it follows that c^* and $T(c^*)$ have the looping coefficient $+1$. Therefore by (44)

$$\text{looping coefficient } (c^*, D(c^*)) = +1. \tag{46}$$

Every two Clifford-circles have a definite spherical distance. If the distance is $\pi/2$ the Clifford-circles are called conjugate. In 4-dimensional $x_1, \cdots, x_4$-space the planes of conjugate Clifford-circles are orthogonal. We deform the mapping D of S^3 into a mapping D_1 which

maps every Clifford-circle into its conjugate. We only have to move $D(P)$ on the arc $PD(P)$ into the point $D_1(P)$ whose distance from P is $\pi/2$. Therefore we have from (46)

$$\text{looping coefficient } (c^*, D_1(c^*)) = +1. \tag{47}$$

This means that D_1 maps every Clifford-circle with degree $+1$ onto its conjugate. On the other hand one can consider the set of the Clifford-circles as a fibre bundle whose base space is a two-sphere S^2. D_1 induces in S^2 a one-to-one mapping without fixed points which consequently has the degree -1. This together with the fact that the fibres are mapped with degree $+1$ implies that D_1 and D have degree -1 in contradiction to our previous statement that the degree of D is $+1$. This proves the following theorem:

THEOREM 4. *A continuous vector field on the 3-sphere which differs sufficiently little from a field of Clifford-parallels and which sends through every point exactly one integral curve has at least one closed integral-curve.*

This result can be generalized. If in our previous notation $P=D(P)$, we say that the streamline through P closes after one circulation. Let c^* be the Clifford-circle through P and T the above-mentioned mapping of the $\beta/2$ neighborhood of c^*. Then T induces in the surface $z=$const. that contains P a mapping with the fixed point P. If P is an isolated fixed point we may assign its index to the closed streamline through P. Suppose that there is only a finite number of streamlines which close after one circulation. Then one can prove by means of Theorem 2′ that the sum of the indices of these streamlines is always equal to 2. More generally, if S^2 is replaced by an arbitrary three-dimensional fibre-bundle, this sum is equal to the Euler characteristic of the two-dimensional base space.

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COMPACT SETS OF FUNCTIONS AND FUNCTION RINGS

DAVID GALE

A widely used theorem of analysis asserts that a uniformly bounded, equicontinuous family of functions has a compact closure in the space of continuous functions. This lemma, variously attributed to Arzela, Escoli, Montel, Vitali, and so on, is of importance in the theory of integral equations, conformal mapping, calculus of variations, and so on. In recent years the lemma has been generalized by S. B. Myers [1].[1] A part of his results may be formulated as follows;

If a topological space X is either (a) locally compact, (b) satisfies the first axiom of countability, and if Y is a metric space, then a family F of continuous functions from X to Y is compact (in a suitable topology) if and only if (1) $F(x)=\bigcup_{f\in F} f(x)$ is compact for all $x\in X$, (2) F is closed, (3) F is equicontinuous.

The main purpose of §1 of this paper is to characterize compact sets of functions when Y is any regular topological space. The problem is therefore to find a condition to replace equicontinuity, which no longer makes sense. We obtain such a characterization which holds for an easily described class of spaces X which includes both locally compact and first countable. In §2 these results are applied to obtain a sort of duality theorem for the ring of real-valued continuous functions, $R(X)$, on a space X. Namely, it is shown that, under quite general conditions, the space X is homeomorphic with the space $H(X)$ of continuous homomorphisms from the ring $R(X)$ onto the real numbers, where $R(X)$ and $H(X)$ are given the "compact-open" topology.

1. Compact sets of functions. All spaces to be considered in what follows will be assumed Hausdorff.

We introduce now the notation of a k-space first defined by Hurewicz.

DEFINITION. A topological space X is called a k-space if $S\subset X$ is closed if and only if S intersects every compact subset of X in a compact subset.

The k-spaces are an important class in that they are exactly the spaces whose topology is determined once the compact subsets are

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

known. Let us verify that k-spaces include both locally compact spaces and spaces satisfying the first axiom of countability.

Let X be locally compact and suppose $S \subset X$ intersects every compact set in a compact set. We must show that S is closed, so suppose $x \notin S$. Let N be a neighborhood of x with compact closure $\overline{N}$. Then by assumption $\overline{N} \cap S$ is compact and does not contain x, so there is a neighborhood N' of x such that $N' \cap (\overline{N} \cap S) = \varnothing$. Letting $N'' = N' \cap N$, we get $N'' \cap S = N' \cap N \cap S \subset N' \cap \overline{N} \cap S = \varnothing$, so S is closed.

Let X be first countable, and suppose $S \subset X$ intersects every compact set in a compact set. If x is in the closure of S, then it is a limit point of a sequence (x_i) of points in S, and the set K consisting of the sequence (x_i) and the point x is compact. By assumption $K \cap S$ is compact hence closed, so since each point of (x_i) lies in $K \cap S$, it follows that $x \in K \cap S$ and hence $x \in S$ and S is closed.

We shall now redefine briefly the notions of compact-open (c.o.) and point-open (p.o.) topologies for function spaces.

DEFINITION. If X and Y are spaces and $K \subset X$ is compact and U is open in Y, we denote by $[K, U]$ the set of all continuous functions from X to Y such that $f(K) \subset U$. We denote by Y^X the space of all continuous functions from X to Y in which the sets $[K, U]$ form a subbase (the compact-open topology).

If $x \in X$ and U is open in Y, we denote by $|x, U|$ the set of all functions (not necessarily continuous) from X to Y such that $f(x) \in U$ and we denote by $|Y^X|$ the space of all functions from X to Y in which the sets $|x, U|$ form a subbase (point-open topology).

Some further notation will be needed.

If $F \subset Y^X$ and $S \subset X$ and $T \subset Y$ we define,

$$F(S) = \bigcup_{f \in F} f(S), \qquad \bigcup F^{-1}(T) = \bigcup_{f \in F} f^{-1}(T), \qquad \bigcap F^{-1}(T) = \bigcap_{f \in F} f^{-1}(T).$$

THEOREM 1. *If X is a k-space and Y is regular, then a set of functions $F \subset Y^X$ is compact if and only if;*

(1) *F is closed in Y^X;*

(2) *$F(x)$ is compact for every $x \in X$;*

(3) *If Q is closed in F and C is closed in Y, then $\bigcup Q^{-1}(C)$ is closed in X.*

The condition (3) can be stated in the clearly equivalent form.

(3′) *If Q is closed in F and U is open in Y then $\bigcap Q^{-1}(U)$ is open in X.*

(This condition represents a sort of simultaneous continuity of the functions Q, requiring that the inverse image of an open set be open in this generalized sense.)

PROOF. We first show the necessity of (1), (2), and (3). If F is

compact, then (1) is satisfied since Y^X is Hausdorff [2]. To prove (2), define for each $x \in X$, ϕ_x from Y^X to Y by $\phi_x(f) = f(x)$. Then ϕ_x is continuous, for if U is open in Y and $\phi_x(f) \in U$, we let $N(f) = [x, U]$. $N(f)$ is open by definition of the topology in Y^X, and clearly $\phi_x(N(x)) \subset U$. Therefore ϕ_x is continuous and hence if F is compact, so is $\phi_x(F) = F(x)$.

Condition (3) is the one which replaces equicontinuity and is less trivial to prove. We observe first that if Q is closed in F, then it is compact, so it suffices to demonstrate (3) for compact sets of functions, hence for F itself.

Let C be closed in Y and set $E = \bigcup F^{-1}(C)$. To show that E is closed, we must show that for any compact set K, $K \cap E = S$ is compact or, what is equivalent, closed. We shall show that if $x_0 \notin S$, then $x_0 \notin \overline{S}$. There are two cases.

Case 1, $x_0 \notin K$. Then there is obviously a neighborhood of x_0 which does not intersect K, hence does not intersect S, so $x_0 \in \overline{S}$.

Case 2, $x_0 \in K$. Then since $x_0 \notin S$, we have $x_0 \notin E$, so $f(x_0)$ is in the complement of $C = U$ for all $f \in F$. Since Y is regular we can chose for each $f \in F$ a neighborhood N_f of x_0 such that $f(\overline{N}_f) \subset U$. Now define $K_f = \overline{N}_f \cap K$. This is a compact set. Finally we define for each $f \in F$, $W_f = [K_f, U]$. These sets are open and cover F, so by compactness we can find functions $f_1, \cdots, f_n$ such that the corresponding W_{f_i} cover F. Thus for every $f \in F$ there is an i such that $f(K_{f_i}) \subset U$, and therefore if $K^* = \bigcap_i K_{f_i}$, we have $f(K^*) \subset U$ for all $f \in F$ so $K^* \cap E = \varnothing$. Now let $N^* = \bigcap_i N_{f_i}$. We show that $N^* \cap S$ is empty, for $N^* \cap S \subset N^* \cap K \subset K^*$, so $(N^* \cap S) \cap E = \varnothing$, and since $S \subset E$, $N^* \cap S = \varnothing$. Therefore S is closed and hence compact as it lies in K, so since X is a k-space we conclude that E is closed.

The proof of sufficiency is quite similar to the proof given by Myers for the equicontinuous case. A preliminary result is needed.

LEMMA. *If a family F in Y^X satisfies condition* (3) *of the theorem, then the compact-open and point-open topologies are equivalent on F.*

PROOF. It is clear that every open set in $|Y^X|$ is also open in Y^X, giving continuity in one direction. It remains to show that the sets $[K, U] \cap F$ are also open in the point-open topology. Let $f \in F$ map K compact into U open in Y. Then for each $x \in K$, choose V_x open so that $f(x) \in V_x \subset \overline{V}_x \subset U$, which is possible since Y is regular. Since $f(K)$ is compact, we choose $x_1, \cdots, x_n$ such that $\bigcup_i V_{x_i} \supset f(K)$ and let $V = \bigcup_i V_{x_i}$. Then $f(K) \subset V \subset \overline{V} \subset U$.

Now for each $x \in K$, define $W_x = \{f' | f' \in F \text{ and } f'(x) \in \overline{V}\}$. W_x is closed and nonempty, containing f. Therefore it follows from con-

dition (3′) that $\bigcap W_x^{-1}(U)$ is an open neighborhood of x, denoted by N_x. The neighborhoods N_x cover K and hence we can find $x_1', \cdots, x_m'$ in K such that $\bigcap_i N_{x_i'} \supset K$.

Now $f \in \bigcap_i |x_i', V|$. We shall show $\bigcap_i |x_i', V| \subset [K, U]$, for suppose $f'(x_i') \in V$ for all i. Then for any $x \in K$ there is an i such that $x \in N_{x_i}$, hence $x \in \bigcap W_{x_i}^{-1}(U)$ and therefore $g(x) \in U$ for all $g \in F$ such that $g(x_i) \in \overline{V}$. But since $f'(x_i) \in V$, it follows that $f'(x) \in U$, hence $f'(K) \subset U$, completing the proof.

We now obtain the sufficiency of the conditions of Theorem 1 without requiring that X be a k-space. Let $F \subset Y^X$ satisfy (1), (2), and (3). Since F satisfies (3), it may be considered as imbedded topologically in $|Y^X|$ by the lemma, and we shall denote its image by F'. Now $|Y^X|$ can be considered as the topological product of Y with itself X times, and since $F(x)$ is compact for each x, we get $Q = \prod_{x \in X} F(x)$ is compact, and since $F' \subset Q \subset |Y^X|$, it suffices to prove that F' is closed in $|Y^X|$.

Suppose then $g' \in \overline{F}'$. The theorem will be proved if we can show g' continuous, for the set $F \cup g$ in Y^X will also satisfy condition (3) and will be homeomorphic to $F' \cup g'$ so that $g \in \overline{F}$. But since F is closed in Y^X, we have $g \in F$ or $g' \in F'$, so F' is closed, hence compact.

We show g' is continuous. Let x_0 be in X and $g(x_0)$ in U open in Y. Choose V open in Y such that $g(x_0) \in V \subset \overline{V} \subset U$ and let $T = \{f | f \in F$ and $f(x_0) \in \overline{V}\}$. T is closed in F and nonempty, since $g' \in Q$ so that $g'(x_0) \in F(x_0)$, and hence for some $f_0 \in F$ we have $f_0(x_0) = g'(x_0) \in \overline{V}$. Hence by condition (3), $\bigcap T^{-1}(V) = N(x_0)$ is open in X. We shall show that $g'(N(x_0)) \subset U$. Let $x \in N(x_0)$ and let W be any neighborhood of $g(x)$ in Y. Since $g \in \overline{F}'$, there exists $f' \in F$ such that $f' \in |x_0, V| \cap |x, W|$. But $f' \in |x_0, V|$ implies $f' \in T$, and hence $f'(x) \in V$ since $x \in N(x_0)$. Hence $W \cap V \neq \varnothing$ for any $W \ni g(x)$, which means $g(x) \in \overline{V}$ and therefore $g(x) \in U$ and g is continuous.

REMARK. Condition (3) of Theorem 1 is not equivalent to equicontinuity when Y is metric. In fact one easily constructs an equicontinuous family from the unit interval to the reals which does not satisfy (3).

We now derive a simple consequence of Theorem 1.

THEOREM 2. *Let F be a family of continuous functions from a k-space X into Y regular. Define $\phi: X \to Y^F$ by $\phi(x)(f) = f(x)$. Then ϕ is continuous.*

PROOF. Let x_0 be in X and let $[K, U]$ be a subbasic neighborhood of $\phi(x_0)$ in Y^F, with K compact in F, U open in Y. This means $K(x_0) \subset U$

or $\phi(x_0) \in \bigcup K^{-1}(U)$. But by Theorem 1, $N(x_0) = \bigcap K^{-1}(U)$ is open in X and, since $\phi(N(x_0)) \subset [K, U]$, ϕ is continuous.

2. **Application to function rings.** Let R denote the real numbers, X a topological space, and $R(x)$ the ring of all continuous functions from X into R in the compact-open topology. A sub-algebra $A(X) \subset R(X)$ will be called separating over X if for $x \neq x' \in X$, there exists $f \in A(X)$ such that $f(x) = 0$, $f(x') \neq 0$.

We define H to be the set of all continuous homomorphisms from $A(X)$ onto R and we define the natural mapping, ϕ, from X into H as follows: $\phi(x)(f) = f(x)$ for all $f \in A(X)$. To justify this definition one must show that $\phi(x)$ is actually a continuous homomorphism of $A(X)$ onto R, which fact, however, is immediately verified.

LEMMA. *If $A(X)$ is separating, then ϕ is one-to-one and onto.*

PROOF. The one-to-one property follows from the fact that A is separating. Namely $\phi(x) = \phi(x')$ implies $\phi(x)(f) = \phi(x')(f)$ for all $f \in A(X)$, or $f(x) = f(x')$ and hence $x = x'$.

To show that ϕ maps onto H, let h be a continuous homomorphism from $A(X)$ onto R and let M be the closed maximal ideal which is its kernel. We shall show that there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in M$, for suppose this is false. Then M is again a separating algebra on X, for given $x \neq x' \in X$ choose $f \in M$ such that $f(x') \neq 0$, and $g \in A(X)$ such that $g(x) = 0$, $g(x') \neq 0$. Then if $k = fg \in M$ we have $k(x) = 0$, $k(x') \neq 0$. Now since M is separating, we can apply the Stone-Weierstrass approximation theorem [3] which states that a separating algebra of continuous functions is dense in $R(x)$ in the c.o. topology. But since M is closed, this would mean $M = A(x)$, contrary to the fact that h maps onto R. The contradiction shows that every function in M vanishes on some point $x_0 \in X$. Since M is maximal, it follows that M consists of all $f \in A(X)$ such that $f(x_0) = 0$. Thus h has the same kernel as the homomorphism $\phi(x_0)$, which means they differ by an automorphism of R, and since the only such automorphism is the identity we conclude that $h = \phi(x_0)$.

The above shows that X and H are in a natural one-to-one correspondence. The question then occurs as to whether one can make ϕ into a homeomorphism by giving H a function space topology. We consider the case where the algebra $A(X)$ is completely regular, which means, for any C closed in X and $x \notin C$, there exists $f \in A(X)$ such that $f(x) \notin \overline{f(C)}$. For this case it is known and easily verified that ϕ is a homeomorphism when H is given the p.o. topology. First,

ϕ is continuous for if $|f, U|$ is subbasic in H, then $\phi^{-1}(|h, U|) = \phi^{-1}\{h \mid h(f) \in U\} = \{x \mid f(x) \in U\} = f^{-1}(U)$, which is open since f is continuous. Also ϕ^{-1} is continuous, for let C be closed in X and suppose $h \notin \phi(C)$. If $x = \phi^{-1}(h)$, then by complete regularity there exists $f \in A(X)$ such that $f(x) \notin \overline{f(C)}$ and if we let U be the complement of $\overline{f(C)}$ it follows that $|f, U|$ is a subbasic neighborhood of h which does not intersect $\phi(C)$, so $\phi(C)$ is closed.

As an application of the results of §1 we prove:

THEOREM 3. *If X is a k-space and $A(X)$ is completely regular, then if H is given the c.o. topology, ϕ is a homeomorphism.*

PROOF. The continuity of ϕ is precisely the statement of Theorem 2. The fact that ϕ^{-1} is continuous is seen since we have observed that ϕ^{-1} was continuous in the p.o. topology and a fortiori continuous in the stronger c.o. topology.

We remark that the above theorem shows that p.o. and c.o. topology are equivalent on H.

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TWO-ENDED TOPOLOGICAL GROUPS

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Introduction. Let G be a locally compact, connected topological group (satisfying the second countability axiom). Let G^* be a compact space which contains a dense subset G' homeomorphic to the space G and is such that G^*-G' is totally disconnected. Then, Freudenthal has proved [1, Satz 1X, p. 277][1] that the set G^*-G' consists of at most two distinct points. Actually, Freudenthal's theorem even for topological groups is more general than here stated, and this theorem is an application to group spaces of a wider theory of "ends" of topological spaces. However, we shall quote only so much of Freudenthal's results as are necessary to this paper.

It will be convenient to regard G' as identical with G so that G is topologically imbedded in G^*. We shall call a locally compact, connected group G *two-ended* if a G^* exists such that G^*-G consists of two distinct points. The simplest example of such a group is the additive group of reals. Other examples are afforded by the direct product of this group and any compact connected topological group; it is likely that these are the only examples.

The principal objective of this note is the following theorem.

THEOREM A. *If a locally compact, connected topological group G is two-ended, then G contains a closed subgroup T isomorphic to the group of reals such that the coset-space G/T is compact; moreover, the space G is the topological product of the axis of reals by a compact connected set homeomorphic to the space G/T.*

1. **Definitions.** Now let G be two-ended and G^* compact, and necessarily connected, such that G^*-G consists of a pair of points. We shall denote one of these by e_L and the other by e_R. The space G^* is not a group, but we may continue to speak of the group product fg when $f, g \in G \subset G^*$. Moreover [1] each $f \in G$ may be regarded as a homeomorphism $f(G^*)=G^*$ by the definitions: $f(e_L)=e_L$, $f(e_R)=e_R, f(g)=fg \in G$ when $g \in G$. To the product of homeomorphisms f and g there corresponds the homeomorphism associated with $fg \in G$. We shall denote by gK (resp. Kg), $g \in G$, $K \subset G$, the set of points gk (resp. kg), where $k \in K$.

The two following properties [1] of G^* are of great importance in this work.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

(1.1) If $g_n \in G$, $n=1, 2, \cdots$, and $g_n \to e_R$, then $g_n^{-1} \to e_L$; and if $g_n \to e_L$ then $g_n^{-1} \to e_R$.

(1.2) If K is a compact subset of G and $g_n \to e_R$ (resp. e_L), then $g_n K \to e_R$ (resp. e_L).

In virtue of the known structure of locally compact abelian groups we may, in any locally compact group, distinguish between two classes of elements [2, Lemma 2, p. 96]. One of these is:

(1.3) The class which we shall denote by C consisting of those element of G which are contained in *compact* subgroups of G. For any elements $c \in C$, and any neighborhood V of the identity of G, there exists an integer n such that $c^n \in V$.

Each element of G not in C generates a subgroup of G isomorphic to the group of integers. For such elements no sequence of distinct powers has any limit in G. Now in a two-ended G we may distinguish further:

(1.4) The class R of elements of G such that if $r \in R$ then $r^n \to e_R$ (in G^*), and

(1.5) The class L such that $l \in L$ if $l^n \in e_L$ (in G^*).

It is clear from (1.1) that R and L are each the set of inverses (in G) of the other, are mutually exclusive, and are homeomorphic.

It is another matter, of course, to assert that none of these classes L, C, and R is empty, or that they comprise all elements of G. This will be shown in the succeeding paragraphs. The assumption that G is two-ended is to be understood throughout; many of the lemmas are not valid in the more general case.

2. The key lemma.

PRINCIPAL LEMMA. *If $g_n \to e_R$, $g_n \in G$, then for all but a finite number, $g_n \in R$.*

PROOF. From the definition of ends [1] it follows at once that there exists a closed compact set $K \subset G$, and open sets A and B of G^* such that

$$G^* = A \cup K \cup B, \tag{2.1}$$

$$e_L \in A,\ e_R \in B,\ A \cap K = K \cap B = B \cap \overline{A} = A \cap \overline{B} = 0.$$

By (1.1) for almost all n, $g_n K \subset B$. Let g denote any g_n for which $gK \subset B$. Then $\overline{A} \cap gK = 0$.

Now let A_0 denote the component of $G^* - K$ which contains e_L. Then A_0 is a maximally connected subset of A, A_0 is closed in A, and $\overline{A}_0 \cap K$ is not empty since G^* is connected [3, p. 16]. Let k denote some element of the set $\overline{A}_0 \cap K$.

$$G^* = g(G^*) = gA \cup gK \cup gB, \tag{2.2}$$

and it is easily seen that gA_0 is the component of G^*-gK which contains $e_L=g(e_L)$. Then, since $gK\subset B$, $\overline{A}_0\cap gK=0$ and $\overline{A}_0\subset gA_0$. Moreover, since $gk\in B$, $gk\notin\overline{A}_0$ and there is some neighborhood V of k such that $gV\cap\overline{A}_0=0$. Since $k\in\overline{A}_0$, $V\cap A_0$ is not empty. Choose some $a\in V\cap A_0$. There is a neighborhood W of the identity such that $Wa\subset V$. Then $gWa\subset gV$, and $(gWa)\cap\overline{A}_0=0$. Thus we have shown that

$$\overline{A}_0 \subset gA_0 - g(Wa\cap\overline{A}_0), \tag{2.3}$$

and if we denote g^{-1} by f, we may write this as

$$f\overline{A}_0 \subset A - Wa\cap\overline{A}_0. \tag{2.4}$$

Now, for every integer n,

$$f^n\overline{A}_0 = f^{n-1}f\overline{A}_0 \subset f^{n-1}\overline{A}_0 \subset \cdots \subset f\overline{A}_0 \subset A - Wa\cap\overline{A}_0. \tag{2.5}$$

In consequence of this, for every n, $f^na\notin Wa$, and therefore $f^n\notin W$.

But now it follows from (1.3) that no sequence of powers of f can converge to any element of G. On the other hand, since $f^na\subset A$, no subsequence of the points f^na, $n=1, 2, \cdots$, can converge to $e_R\in B$. Because of (1.2), no subsequence of the set f^n, $n=1, 2, \cdots$, can converge to e_R. Finally, then, $f^n\to E_L$, and in consequence of this $g^n\to e_R$, since $g=f^{-1}$. This proves the lemma.

COROLLARY. *There exists an open set $O^*\subset G^*$, $e_R\in O^*$, such that if $g\in O^*\cap G$, then $g\in R$.*

This follows from the lemma by the observation that if no such O^* existed one could construct a sequence $g_n\to e_R$ such that $g_n\notin R$, and this would contradict the lemma.

LEMMA. *If $g\in G$ and for some positive integer k, $g^k\in R$, then $g\in R$*

Let Q denote an arbitrary open subset of G^* containing e_R. Since $g^k\in R$, the sequence $g^{nk}\to e_R$, k fixed. Therefore for each $m=0, 1, 2, \cdots, k-1$, the sequences $g^{nk}g^m=g^{nk-m}\to e_R$ (k and m fixed) and there exists an integer N such that for all $n\geqq N$ and all $m=0, 1, \cdots, k-1$, $g^{nk+m}\in Q$. This means, of course, that in the sequence $g, g^2, g^3, \cdots$, all but a finite number belong to Q. Since Q is an arbitrary neighborhood of e_R, this implies that $g^n\to e_R$ and, consequently, $g\in R$.

COROLLARY. *R is open, and $R\cup e_R$ is open.*

Let $g\in R\subset G$. Then for some integer k, $g^k\in O^*\cap G$, for the O^* of the

preceding corollary. The set $O^* \cap G$ is open and contained in R. Then there is a neighborhood V of g such that $V^k \subset O^* \cap G$. Now for any $f \in V$, $f^k \in O^* \cap G \subset R$ and consequently $f \in R$. This establishes that R is open and, since $O^* \subset R \cup e_R$, it is clear that $R \cup e_R$ is open.

3. **Another lemma.** We have proved now that R is open and not empty. By symmetrical arguments, or by an application of (1.1), the set L is not empty and open. Moreover, in view of the preceding corollary, $e_L \cup L \cup R \cup e_R$ is an open set whose complement clearly is a closed and *compact* subset of G. Further, if c is an element of this complement, then $c^n \notin L \cup R$ for every integer n, by a preceding corollary. Then $c \in C$ as defined in (1.3). Since $R \cap L = 0$, and since G is connected, it follows that

$$G = L \cup C \cup R, \tag{3.1}$$

where C is closed, compact, not empty. We observe, from the form of (3.1), that C separates G^* between e_L and e_R; that is, there is no connected subset of $G^* - C$ which contains both e_L and e_R.

LEMMA. *The identity e of G is a limit of element of R.*

We know that $C \cap \overline{R}$ is not empty, since G is connected. Let $c \in C \cap \overline{R}$, and let V be an arbitrary neighborhood of the identity. There exists an integer n such that $c^n \in V$, by (1.3), and there must exist a neighborhood W of c such that $W^n \subset V$. This implies that there are in V elements of the form r^n, $r \in W \cap R$. For such r, $r^n \in R$. Therefore $R \cap V$ is not empty, and since V is an arbitrary neighborhood of the identity we have established the lemma.

4. **One-parameter subgroups.** It is our next task to construct in G a one-parameter group T which is *closed* in G. Now if T denotes any one-parameter subgroup of any locally compact group G, then by a result of which we have already taken partial advantage [2, p. 96] *either* T belongs to some compact subgroup of G, and in that case $T \subset C$, *or* T *is* a closed subgroup of G. Therefore, to construct in G a subgroup isomorphic to the additive group of reals it suffices to construct a one-parameter group which contains an element not in C.

LEMMA. *G contains a connected abelian group A which contains an element of R.*

The group A will be constructed with the aid of the preceding lemma. Let $r_n \to e$, $r_n \in R$. Denote by Q an open compact subset of G such that $C \subset Q$, and denote by A_n the group generated by r_n. This

group is isomorphic to the additive group of integers. Then A_n will have elements in Q and also in the complement of $\overline{Q}$, and it is easy to see that there is a subsequence A_{n_i} of the sequence A_n such that some point of $\overline{Q}-Q$ is in the sequential limit set of the sequence of sets A_{n_i}. This sequential limit set is the desired group A. We have merely sketched the argument because it occurs in substantially the form needed here in [4, Lemma 5, p. 112].

COROLLARY. *G contains a subgroup T isomorphic to the additive group of reals.*

The closure of the group A of the preceding lemma is a locally compact connected abelian group and is therefore the direct product of a compact subgroup by an n-dimensional vector group [5]. Since $\overline{A}$ cannot be compact, containing an element of R, it follows that the vector group is not the identity and must contain a subgroup T of the desired property. This is a closed subgroup of G, as well as of $\overline{A}$.

5. **Conclusion of the proof of Theorem A.** We are nearing the end of our task. We know that G contains at least one subgroup isomorphic to the reals. Suppose that T is any such closed one-parameter subgroup. Then the set $e_L \cup T \cup e_R$ is a closed subset of G^* homeomorphic to a simple arc with end points e_R and e_L. Moreover, this is also true of the set $e_L \cup gT \cup e_R$ for every $g \in G$. Then it follows from the remark following (3.1) that every coset gT contains at least one point of C. Let us now consider the coset-space, for definiteness the right coset space G/T. Since T is closed in G, G/T may be topologized and is a locally compact topological space [5]. It may assist the reader to remark that an essential feature of this topology is the fact that if $g_n \to g$, where g, $g_n \in G$, and if the sequence $g_n t_n$, $t_n \in T$, has any limit point in G then this limit point is of the form gt for some $t \in T$.

LEMMA. *The coset-space G/T is compact and homeomorphic to a space C^* which is a continuous image of C.*

Let us denote by C_g the set $C \cap gT$, for $g \in C$. This is a closed set. Suppose now that $g_n \to g$, $g_n \in C$, and consider the associated sets $C_n = C \cap g_n T$. Let $k_n \in C_n$. Then $k_n = g_n t_n$, $t_n \in T$, $g_n \in C$. Since C is compact, and $k_n \in C$, it is an easy consequence that the set of t_n, $n = 1, 2, \cdots$, is compact. From this it is clear that any limit point of the set k_n, $n = 1, 2, \cdots$, is of the form $gt \in C_g$, for some $t \in T$. Consequently the sets C_g, $g \in C$, give rise to an upper semi-continuous decomposition [3] of C, and there is a compact space C^* which is

a continuous image of C, $\phi(C)=C^*$. For each $c^*\in C^*$, the set $\phi^{-1}(c^*)$ is some decomposition set $C_g=C\cap gT$.

Now it is clear that each point of C^* is associated uniquely with a coset gT and therefore with a point of the coset-space G/T. Since each coset intersects C, this correspondence extends over all of G/T and all of C^*. From the definition, above, of C^* and the topology of G/T it follows easily that this correspondence is a homeomorphism. This establishes the major part of Theorem A.

The fact that the coset-space G/T is homeomorphic to a compact set C^* is not alone sufficient to insure that G may be expressed as the topological product of C^* and T. This would imply, and is implied by, the existence in G of a closed set $\bar{C}$ meeting each coset gT in one and only one point. But the fact that T is a closed one-parameter group is enough to insure the existence of such a "cross-section." This was proved by Montgomery and the author [6] in a somewhat more general context, and depends on certain local "sections" of regular families of curves constructed by Whitney. We may conclude, then, that G is representable as the direct product of two closed subsets, the set T and a cross-sectioning set $\bar{C}$ which is of necessity homeomorphic to C^*. Since G is connected, it is clear that $\bar{C}$ and C^* are connected as well as compact. This concludes the proof of Theorem A.

6. **An application.** As an application of Theorem A, let us suppose that a two-ended group G possesses a connected subgroup H which is not compact. Then $\overline{H}$ is locally compact and connected and also two-ended. Therefore, $\overline{H}$ contains a subgroup T isomorphic to the group of reals. Since T is in G, it is evident from the preceding section that G/T is compact. All the more, then, $G/\overline{H}$ is compact.

It is an immediate consequence of this remark that if a locally compact connected group G contains a non-compact connected subgroup H such that the coset space $G/\overline{H}$ is *not* compact, then G cannot be a two-ended group. Therefore, by Freudenthal's theorem which forms the point of departure for this note, G can have only a single "end." This sufficient condition for "one-endedness" generalizes a result due to Freudenthal, appearing in an appendix to his work [7] on the "ends" of discrete-spaces.

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THE RESISTANCE OF AN ELECTRICAL NETWORK

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1. **Introduction.** We consider a two-terminal electrical network[1] N of branches $w_1, w_2, \cdots, w_n$, having the independently variable resistances $r_1, r_2, \cdots, r_n$ respectively. The resistance R measured between the terminals of this network is a homogeneous rational function of the r_i. We shall show that R has these properties:

Property A. The first partial derivatives of R are squares of rational functions; in fact, for each m from 1 to n

$$\partial R/\partial r_m = i_m^2, \tag{1}$$

where i_m is the current flowing in w_m when unit current enters the network.

Property B. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be any complex numbers. Let P be a permutation of $1, 2, \cdots, n$, and denote $P(j)$ by i_j. Let r_{i_j} approach α_{i_j} for the successive values $j=1, \cdots, n$. Then R approaches a limit[2] which is independent of P.

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[1] For our purposes we do not consider the power source, or the branches joining it to the terminals, to be part of the network. We assume that the network is passive and contains no transformers, though the latter restriction could be dispensed with. We speak of the resistances of the branches, but since we allow these quantities to assume all complex values, the branches may contain impedances.

[2] The point at infinity of the complex plane may be the limit of R or may be assigned as the value of an α_i. We shall describe in detail the process of obtaining the limit, including the conventions which apply to the point at infinity.

Let us substitute successively the α_{i_j} for the r_{i_j} in numerator and denominator. We assume first that none is infinite. If the denominator does not vanish after all substitutions are made, the resulting constant value is evidently the limit of R. If at some stage in the course of the substitutions the denominator vanishes identically in the remaining r_{i_j}, while the numerator becomes a nonzero function of them, we agree to call infinity the limit of R. Let us suppose both numerator and denominator become identically zero after the kth substitution, while neither was zero before. Then after $k-1$ substitutions $R=U(A'+B'r_{i_k})/V(M'+N'r_{i_k})$, where A', B', M', N' are constants, $B'N' \neq 0$, and U and V are free of r_{i_k}. As $r_{i_k} \to \alpha_{i_k}$ this function approaches $(U/V)B'/N'$. We find the limit of R by taking the limit of U/V and multiplying by B'/N'.

Now consider the case that some α_{i_k} is infinite, and that a limit is not obtained on account of the vanishing of the denominator only, before the substitution of α_{i_k}. Then if r_{i_k} occurs only in the denominator of the function resulting from the earlier substitutions, the limit of R is zero. If it occurs only in the numerator, we agree to call the limit of R infinity. If it occurs in neither we can disregard it and continue substitutions. Finally, if it occurs in both, we may find the limit of R by omitting from the numerator and denominator the terms free of r_{i_k}, cancelling r_{i_k} from the remaining terms, and continuing the substitutions.

It may seem unlikely that Property A can be new.[3] It is intimately related to and easily derived from several well known circuit relations.[4] Nevertheless equation (1) itself does not appear to be known.

We shall give a proof of Property A which we believe will prove instructive without demanding previous acquaintance with network theory.[5] We shall derive the necessary preliminary results since this will require but little more space than is needed to state them accurately.

2. **Preliminary results.** We begin by writing the equations of Kirchhoff for the network N with unit current entering at the terminals. i_k denotes, as in the statement of Property A, the current which then flows in w_k. Let the network contain s nodes or vertices, and t non-self-intersecting loops. We designate the nodes by $v_1, v_2, \cdots, v_s$, where, in particular, v_1 is the node at which current enters the network, and v_s the node at which it leaves. We designate the loops by $l_1, l_2, \cdots, l_t$. We assign a positive direction arbitrarily in each branch w_i and each loop l_i. Then the Kirchhoff equations may be stated as follows:

$$\sum_{k=1}^{n} \epsilon_{jk} i_k = \alpha_j, \qquad j = 1, 2, \cdots, s, \tag{2}$$

$$\sum_{k=1}^{n} \eta_{jk} i_k r_k = 0, \qquad j = 1, 2, \cdots, t, \tag{3}$$

where the α_j, ϵ_{jk}, and η_{jk} have the following values. $\alpha_1 = 1$, $\alpha_s = -1$, $\alpha_j = 0$, $j \neq 1, s$. ϵ_{jk} is $+1$ if the kth branch terminates at the jth node, and its positive direction leads to that node, -1 if the kth branch terminates at the jth node, and its positive direction leads from that node, 0 if the kth branch does not terminate at the jth node. $\eta_{jk} = +1$

[3] Since this was written I have seen a proof of a result almost identical with Property A in Montgomery, Dicke, and Purcell, *Principles of microwave circuits*, McGraw-Hill, 1948, p. 98. It is derived by means of the reciprocity theorem, and used to prove Foster's theorem. There has also appeared a preliminary report of a paper by R. Bott and R. J. Duffin containing, among other results, a proof that Property A holds in a more general situation, and a proof of a partial converse, Bull. Amer. Math. Soc. Abstract 55-3-173.

[4] See, for example, Guillemin, *Communication networks*, Wiley, 1935, vol. 2, pp. 137–139, especially equation 297 on page 139. My attention was called to this equation by Mr. H. W. Becker, with whom I have had a valuable correspondence concerning the problems discussed in this paper.

[5] In the following discussion I have been guided by a paper of H. Weyl, *Repartición de corriente en una red eléctrica*, Revista Matemática Hispano-Americana vol. 5 (1923) pp. 153–164. Our proofs of equations (4) and (6) below are specializations of proofs given there.

if the kth branch is in the jth loop, and their positive directions coincide, -1 if the kth branch is in the jth loop, and their positive directions are opposed, 0 if the kth branch is not in the jth loop.

It has been shown that the system of equations (2) and (3), which need not be independent, possesses a unique solution for the i_k as functions of the r_k.[6] If we wish to consider the network as a purely mathematical entity, they may be taken as defining equations for the currents.[7] It is evident that a change of assignment of positive direction in a branch changes the sign of the current in that branch, and affects no other current.

We define $e_k = r_k i_k$ to be the voltage across a branch w_k in the positive direction. The voltage of a vertex v_i with respect to a vertex v_j is defined as $E_{ij} = \sum \gamma_k e_k$, where the sum is taken along a path running from v_j to v_i, and γ_k is $+1$ or -1 according to whether w_k is traversed in the positive or negative direction while describing the path. The value of E_{ij} is independent of the path chosen because of equations (3).

We agree to write E_i for E_{i0}. Thus $E_0 = 0$. The resistence R is now defined to be E_1. R is independent of the assignment of positive directions in the branches.

We shall now prove the equation

$$R = \sum_{k=1}^{n} i_k^2 r_k. \tag{4}$$

This is the well known equation which states the physically obvious fact that the power consumed by N is the sum of the powers consumed by its branches.

We let v_{h_1}, v_{h_2} be the vertices of w_k, and, in particular, choose the notation so that w_k is described positively in going from v_{h_1} to v_{h_2}. Then $i_k^2 r_k = e_k i_k = (E_{h_2} - E_{h_1}) i_k$, so that

$$\sum_{k=1}^{n} i_k^2 r_k = \sum_{k=1}^{n} (E_{k_2} - E_{k_1}) i_k = \sum_{j=1}^{s} \left(\sum_{k=1}^{n} \epsilon_{jk} i_k \right) E_j. \tag{5}$$

The double sum is obtained by grouping terms in the preceding

[6] More precisely, this means that there exists a unique solution if we regard the r_k as indeterminates in the sense of abstract algebra. A proof of this result is contained in the discussion given by F. J. Murray, *Theory of mathematical machines*, King's Crown Press, New York, 1947, pp. 19–20. In the paper referred to above, Weyl proves that there exists a unique solution for the i_k whenever the r_k are all positive, from which also our statement can easily be deduced.

[7] It is evident that a network can be defined, and equations (2) and (3) formulated, using only the language of combinatorial topology. For our purposes it can lead to no difficulty to use the more familiar physical picture and avoid abstract terms. We shall assume, as we obviously may without loss of generality, that our networks are connected, and that each branch terminates at two distinct vertices.

sum containing the same E_j; and it is easily seen that the coefficients ϵ_{jk} answer the description given after (3). Thus the coefficients of E_j are zero except for $j=1$ or s. Since also $E_s=0$, we have $\sum_{k=1}^{n} i_k^2 r_k = E_1 = R$. This is (4).

Let the functions h_k of the r_i have the property that $\sum_{k=1}^{n} \epsilon_{jk} h_k = 0$, $j=1, 2, \cdots, s$, ϵ_{jk} as in (2). We shall show that

$$\sum_{k=1}^{n} r_k i_k h_k = 0. \tag{6}$$

There is no physically obvious interpretation of (6) as there is of the similar equation (5). However, it may be shown that (6) is equivalent to the minimum dissipation theorem of Maxwell.[8] To prove (6) we proceed as in the proof of (5). We have $r_k i_k h_k = e_k h_k = (E_{k_2} - E_{k_1}) h_k$. Then $\sum_{k=1}^{n} r_k i_k h_k = \sum_{k=1}^{n} (E_{k_2} - E_{k_1}) h_k = \sum_{j=1}^{s} (\sum_{k=1}^{n} \epsilon_{jk} h_k) E_j = 0$, on account of the defining properties of the functions h_k. Thus (6) is proved.

3. **Proof of Property A.** Having obtained these preliminary results, the proof that R has Property A is almost immediate. Let m be a fixed integer, $1 \leqq m \leqq n$. On differentiating (2) with respect to r_m, we see that the functions $\partial i_k / \partial r_m$, $k=1, 2, \cdots, n$, have the property required of the h_k. Thus it follows that

$$\sum_{k=1}^{n} r_k i_k \partial i_k / \partial r_m = 0. \tag{7}$$

If we differentiate (4) with respect to r_m we obtain, using (7),

$$\partial R / \partial r_m = 2 \sum_{k=1}^{n} r_k i_k \partial i_k / \partial r_m + i_m^2 = i_m^2, \tag{8}$$

and this is (1). This completes the proof.

4. **Discussion of Property B.** We turn now to property B. We may note that this property appears to have an obvious intuitive interpretation. For, let the r_i approach the α_i in any order. The resistance R may be expected to vary continuously with the r_i so that its value for the α_i should be obtained in the limit. But the same network results, and therefore the same R should be obtained, regardless of the order of approach of the r_i to the α_i. Now this is precisely what Property B asserts. Nevertheless the functions i_k, as is easily seen, do not possess Property B, and even R does not have a unique limit when the r_i are allowed to approach the α_i along any path.[9]

[8] Maxwell, Clerk, *A treatise on electricity and magnetism*, Clarendon Press, 1873, p. 337.

[9] In fact, if R possessed a unique limit under all circumstances it would be analytic at all points, and the i_k, as the square roots of its derivatives, would also

The function R is known to be linear-fractional. We shall prove Property B by showing that it is a consequence of Property A for linear-fractional functions. In fact we shall prove the following result on the equivalence of these properties:

THEOREM. *Let R be a function of $r_1, \cdots, r_n$ which is linear-fractional in each r_i. Then R possesses Property* A *if and only if it possesses Property* B.

5. **Proof that A implies B.** We shall first prove that Property A implies Property B. Indeed we shall show that if $n-1$ first partial derivatives of the linear-fractional function R are squares of rational functions, then R possesses property B.

We shall use an inductive method of proof. A function of one variable necessarily has property B. We assume therefore that it has been demonstrated that functions of $n-1$ variables, linear-fractional in each, which have $n-2$ derivatives that are squares of rational functions, have Property B. On this basis we proceed to prove our statement concerning R. We need consider only the case that R effectively involves n variables.

We may assume that the α_i we are dealing with are all zero. For otherwise the substitution $r_i' = r_i - \alpha_i$, if α_i is finite, $r_i' = 1/r_i$, if α_i is the point at infinity of the complex plane, transforms R into a function R', linear-fractional in each r_i', whose first partial derivatives with respect to the r_i' are squares of rational functions. The limit of R' as the r_i' approach zero in any order equals the limit of R as the r_i approach the α_i in the same order.

Let us consider the limits approached by R when a particular r_i, which we shall denote by r, is allowed to approach zero first, and the remaining r_i then approach zero successively in any order.

We write $R = (A+Br)/(M+Nr)$, where A, B, M, and N are free of r. Evidently A and M cannot both be identically zero, for then R would be independent of r. If M is identically zero, R approaches infinity under the circumstances considered. If M is not identically zero, A/M is a linear-fractional function of $n-1$ variables of which at least $n-2$ first partial derivatives are squares of rational functions. By the inductive hypothesis A/M approaches a unique limit when its variables approach zero in any order. But the limits obtained for R, after r is set equal to zero, are the limits of A/M. Thus, in any case, R approaches a unique limit when r is the first variable to

possess unique limits. A simple example of the phenomenon under consideration is afforded by the resistance $r_1r_2/(r_1+r_2)$ of two wires in parallel. As r_1 and r_2 approach zero, this function approaches zero, unless r_1 and r_2 vary along a curve tangent to the line $r_1+r_2=0$. In this exceptional case one may obtain any desired limit by choosing an appropriate curve.

approach zero, and the remaining variables approach zero in any order.

Let r, s be any two distinct r_i. To complete the proof it will evidently suffice to consider the limit approached by R as its variables approach zero in the orders $r, s, \cdots$, and $s, r, \cdots$, the ordering of the remaining r_i being immaterial.

We write $R=(A+Br+Cs+Drs)/(M+Nr+Ps+Qrs)$, A, B, C, D, M, N, P, and Q free of r and s. If A and M are not both zero, we see as before that R approaches a unique limit in either of the indicated cases. If A and M are both zero, we find by computing $\partial R/\partial s$ that $(NC-BP)r+(ND-BQ)r^2$ is the square of a polynomial. This implies that $NC=BP$. Now B and N are not both zero. For, if they were, s would not appear effectively in R. Then if N vanishes identically, P vanishes. Similarly if P vanishes, N vanishes. We see then that either P and N are both identically zero, while B and C are not, or $B/N=C/P$.

In the former case R evidently approaches as limit the point at infinity of the complex plane, no matter in which order r and s approach zero. In the latter case the limit of R is the limit of B/N, if s approaches zero first, and it is the limit of C/P if r approaches zero first. But these two limits are equal. Thus the limit of R is independent of the order of approach of the variables to zero in all cases. The first implication holds.

6. Proof that B implies A. We shall use the following lemma to prove that Property A is a consequence of Property B:

Let P be a polynomial in variables $x_1, \cdots, x_n$, quadratic in each of them. If the vanishing of P implies the vanishing of each $\partial P/\partial x_k$, $k=1, \cdots, n$, then P is the square of a polynomial.

Since the lemma is easily verified for polynomials in one variable, we may proceed inductively, assuming the truth of the lemma for polynomials in $n-1$ variables, and proving it for P.

P is a polynomial which vanishes only when all the derivatives $\partial P/\partial x_k$, $k=1, 2, \cdots, n$, vanish. We must prove that it is the square of a polynomial.

If $P=QR$, where Q and R are each polynomials in one or more variables, and have no variables in common, we see that Q and R each satisfy the condition of the lemma. Then by the inductive hypothesis both Q and R are squares of polynomials. We may thus assume that P has no factorization of this sort. Evidently we may also assume that P involves each x_i effectively.

We write $P=A+Bx_n+Cx_n^2$, A, B, and C free of x_n. It is easy to see that $C\neq 0$. From the fact that $\partial P/\partial x_n$ vanishes whenever P does, we deduce that $B^2=4AC$. Let T be the greatest common divisor of

A and C. We put $A=A_1T$, $C=C_1T$. Then $B^2=4A_1C_1T^2$. Since A_1 and C_1 have no factors in common each must be the square of a polynomial. Let $A_1=U^2$, $C_1=V^2$. We have $B=2UVT$, $P=(U+Vx_n)^2T$.

No variable occurring in T can occur in $U+Vx_n$, for then it would appear to at least the third power in P. Because of the limitation assumed on the factorization of P we see that either T or $U+Vx_n$ is a constant. But $U+Vx_n$ cannot be a constant, for then P would be free of x_n. It follows that T is a constant, so that P is the square of a polynomial. Thus the lemma is proved.

We now return to the proof of the theorem. We introduce the notation $R=A/D$, where A and D are polynomials in the r_i without common factors. We use subscript k to denote partial differentiation of A or D with respect to r_k. We also write $A^{(k)}$, $D^{(k)}$ for $A-A_kr_k$, $D-D_kr_k$, respectively. These polynomials are evidently independent of r_k.

We assume that R fulfills the requirement B. We must prove that the partial derivatives of R are perfect squares. This will be accomplished if we can show that the polynomials AD_k-DA_k, $k=1, \cdots, n$, are perfect squares. Now it is easily verified that AD_k-DA_k is a polynomial quadratic in the r_i and free of r_k. In fact we see that

$$AD_k - DA_k = A^{(k)}D_k - D^{(k)}A_k, \qquad k = 1, \cdots, n. \tag{9}$$

We choose some fixed k. Consider some set of values of the r_i other than r_k, say, $r_i=s_i$, $i\neq k$, which annul AD_k-DA_k. We know from the lemma that it will suffice to show that each derivative $\partial(AD_k-DA_k)/\partial r_j$, $j\neq k$, vanishes for these values. For this purpose we assign a value s_k to r_k as follows:

(a) If neither A_k nor D_k is annulled by the s_i, $i\neq k$, we let $s_k=-A^{(k)}/A_k$, noting that $A^{(k)}/A_k=D^{(k)}/D_k$, when the s_i are substituted for the r_i.

(b) If D_k vanishes when the r_i, $i\neq k$, are replaced by the s_i, but A_k does not, we let $s_k=-A^{(k)}/A_k$; and note that $D^{(k)}$ must also vanish.

(c) If A_k vanishes when the r_i, $i\neq k$, are replaced by the s_i, but D_k does not, we let $s_k=-D^{(k)}/D_k$, and note that $A^{(k)}$ must also vanish.

(d) If both A_k and D_k vanish when the r_i, $i\neq k$, are replaced by the s_i, we assign no value to r_k.

We observe that in the first three cases both A and D vanish when the r_i are replaced by the s_i, $i=1, \cdots, n$.

We shall now show that $A_kD_j-D_kA_j$, where j is a positive integer not exceeding n and distinct from k, vanishes when the r_i are replaced by the s_i. In case (a), A and D vanish. The unique limit of A/D is given by A_k/D_k, and also by the limit of A_j/D_j. If D_j does not vanish, we must have $A_k/D_k=A_j/D_j$, after the substitution of the s_i. If D_j vanishes, then A_j must vanish too, in order that the

limit of A_j/D_j may equal the finite value of A_k/D_k. Under either circumstance our statement concerning $A_kD_j-D_kA_j$ is obviously true. In case (b), A and D again vanish, and this again implies that A_k/D_k and A_j/D_j both have the same limit. But this means that D_j as well as D_k must be annulled by the s_i. Similarly, in case (c) both A_j and A_k vanish. Thus in either of these cases our statement is true. In case (d) it is obvious.

With these observations we can now complete the proof that $\partial(AD_k-DA_k)/\partial r_j$ is annulled by the s_i. We have,

$$\frac{\partial}{\partial r_j}(AD_k - DA_k) = (AD_{kj} - DA_{kj}) + (A_jD_k - D_kA_j). \tag{10}$$

We have just shown that the term in the second bracket on the right is annulled by the s_i. In cases (a), (b), and (c) the first bracket is known to vanish since A and D are annulled. It remains only to prove that the first bracket vanishes also in case (d).

We may assume that the polynomials A, D are not both zero, and also that A_{kj}, D_{kj} are not both zero after the r_i are replaced by the s_i. We write

$$A/D = (E + Fr_j + Gr_k + Hr_jr_k)/(M + Nr_j + Pr_k + Qr_jr_k),$$

E, F, G, H, M, N, P, Q free of r_j and r_k. Evidently $H=A_{kj}$, $Q=D_{kj}$. We see that $G+Hr_j=P+Qr_j=0$, when the r_i are replaced by the s_i. It follows that $E+Fr_j$, $M+Nr_j$ are not both zero after these replacements. Finally H and Q are not both zero after the substitutions.

We consider the limit of R as r_k approaches infinity and the remaining r_i approach s_i. If r_k approaches infinity before the other variables approach their limits, we obtain $\lim R=\lim (G+Hr_j)/(P+Qr_j)$, and this is the value assumed by H/Q when the r_i are replaced by s_i, or infinity if $Q=0$.[10] If we let the r_i approach s_i before r_k approaches infinity, we have that R approaches the value taken by $(E+Fr_j)/(M+Nr_j)$ when the r_i are replaced by s_i, or that R approaches infinity if $M+Nr_j$ becomes zero after these substitutions. The equality of the limits now shows that either $(E+Fr_j)/(M+Nr_j) = H/Q$, or $M+Nr_j=Q=0$ after the replacements. In either case we have $(E+Fr_j)Q=(M+Nr_j)H$, and therefore $(E+Fr_j+Gr_k+Hr_jr_k)Q - (M+Nr_j+Pr_k+Qr_jr_k)H=0$ after the replacements. But the left-hand side of this expression is precisely $AD_{kj}-DA_{kj}$.

[10] For the limit may be computed by substituting the s_i for the r_i in any order. Let the substitution of s_j for r_j be made last. If Q vanishes, then so does P, but not H, and we obtain, at the step before replacement of r_j, the value infinity for the limit. If Q does not vanish, L'Hôpital's rule, which coincides with the procedure of footnote 2, gives the stated value.

We have thus seen that in all cases $\partial(AD_k - DA_k)/\partial r_j$ is annulled by the s_i. The lemma now shows that $AD_k - DA_k$ is a perfect square. This completes the proof that Property B implies Property A.

COROLLARY. *If a function R, linear-fractional in its n variables $r_1, r_2, \cdots, r_n$, has $n-1$ first partial derivatives which are squares of rational functions, then it possesses Property* B, *and all its first partial derivatives are squares of rational functions.*

Since Property B is implied by the fact that $n-1$ first partial derivatives are perfect squares, and since it in turn implies Property A, the corollary follows immediately.

7. **Conclusion.** It was our original hope that Property A, or its equivalent B, together with the obvious requirements

(a) that R be linear homogeneous,
(b) that it be linear-fractional in each r_i,
(c) that the coefficients of each term in the numerator and denominator be $+1$ after all common factors have been removed,

would be necessary and sufficient conditions for R to be the resistance of a two-terminal network composed of the n branches $w_1, w_2, \cdots, w_n$, having the independently variable resistances $r_1, r_2, \cdots, r_n$ respectively. However these conditions are not sufficient as the following considerations show.[11]

Let R satisfy the conditions stated above. Let $s_i = 1/r_i$, $S = 1/R$. Then S, as a function of the s_i, satisfies these conditions, and is termed the dual of R. To any planar[12] network N there exists a conjugate network N' which contains a vertex corresponding to each loop, and a loop corresponding to each vertex of N. The resistance of N' is the dual of the resistance of N. Nonplanar networks possess no conjugates, so that one is led to surmise that the duals of their resistances are not the resistance of any network. We may construct a nonplanar network by joining six branches in the form of a hexagon and connecting two additional branches between pairs of opposite vertices. Current enters and leaves at the remaining vertices. Our surmise is easily verified for the dual of the resistance of this network. Thus this dual is a function meeting the stated conditions, yet not the resistance of any network.

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[11] A converse result of this type has been obtained, however, in a more general situation by R. Bott and R. J. Duffin; see footnote 3. We are indebted to Mr. Becker for the counterexample given below.

[12] A network, in the sense defined in (1), is planar if it can be represented in a plane, together with a branch joining its terminals, in such a manner that no branches meet except at their common vertices.

ON PARALLEL HYPERSURFACES IN THE ELLIPTIC AND HYPERBOLIC n-DIMENSIONAL SPACE

L. A. SANTALÓ

1. **Introduction.** Let S^{n-1} be a hypersurface of class C^2 in the elliptic or hyperbolic n-dimensional space, which is closed and bounding and whose principal curvatures with respect to an inside normal are all positive. Let $S^{n-1}(\lambda)$ be the hypersurface parallel to S^{n-1} at distance λ.

If $\rho_1, \rho_2, \cdots, \rho_{n-1}$ are the principal radii of curvature of S^{n-1} at a point P and dP denotes the element of area at P, the mean curvatures of S^{n-1} are defined by

$$M_i = \int_{S^{n-1}} \left(\sum \frac{1}{\rho_{r_1}\rho_{r_2}\cdots\rho_{r_i}} \right) dP, \qquad i = 0, 1, \cdots, n-1, \tag{1.1}$$

where the sum is extended to the $C_{n-1,i}$ combinations of ith order of the indices $1, 2, \cdots, n-1$. In particular, M_0 coincides with the area A of S^{n-1}.

Herglotz [6][1] and, from a more general point of view, Allendoerfer [1] have obtained the area $A(\lambda)$ and volume $V(\lambda)$ of the parallel hypersurface $S^{n-1}(\lambda)$, which can be expressed as linear functions of the mean curvatures M_i of S^{n-1} with coefficients depending upon λ. For this purpose it is enough to find the expression of $A(\lambda)$, that is, $M_0(\lambda)$, because $V(\lambda)$ is then given by

$$V(\lambda) = V + \int_0^\lambda A(\lambda)d\lambda. \tag{1.2}$$

The purpose of the present note is to extend these results to the evaluation of all mean curvatures $M_i(\lambda)$ of $S^{n-1}(\lambda)$. The resulting formulae are also linear with respect to M_i; they are (2.9) for the elliptic case, and (3.2) for the hyperbolic case. For $i=0$, they give the value of $A(\lambda)$ obtained by Herglotz and Allendoerfer.

As a consequence, in the elliptic case we obtain the relation (4.2) between the mean curvature of an S^{n-1} and those of its polar hypersurface. Finally we obtain the equations (5.3) which hold for the mean curvatures of convex surfaces of "constant width" in the elliptic or hyperbolic n-dimensional spaces.

In all these questions, in order to obtain simplifications in the re-

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

sulting formulas the generalized Gauss-Bonnet formula as obtained by Allendoerfer-Weil [2] plays a fundamental role. In our particular case of the elliptic and hyperbolic space, this formula can be written (see [1]):

For $n-1$ even

$$C_{n-1}M_{n-1} + C_{n-3}M_{n-3} + \cdots + C_1M_1 + K^{n/2}V = -\omega^n\chi'/2 \tag{1.3}$$

and for $n-1$ odd

$$C_{n-1}M_{n-1} + C_{n-3}M_{n-3} + \cdots + C_0M_0 = \omega^n\chi'/2 \tag{1.4}$$

where ω^j is the surface area of a j-dimensional unit sphere ($\omega^0 = 2$) and

$$C_i = \frac{\omega^n}{\omega^i\omega^{n-1-i}} K^{(n-1-i)/2},$$

being $K=1$ in the elliptic and $K=-1$ in the hyperbolic case. χ' is the inner characteristic of the volume bounded by S^{n-1}; if S^{n-1} is a topologic sphere it is $\chi' = -1$ for $n-1$ even and $\chi' = 1$ for $n-1$ odd.

2. **The elliptic case.** Let C_i ($i=1, 2, \cdots, n-1$) be the lines of curvature of S^{n-1} which pass through the point P and let ds_i be the element of arc of C_i at P. The element of area of S^{n-1} at P will be

$$dP = ds_1 ds_2 \cdots ds_{n-1}. \tag{2.1}$$

If ρ_i is the principal radius of curvature at P corresponding to C_i and R_i represents the distance from P to the contact point of the normal to S^{n-1} at P with the envelope of the normals to S^{n-1} along C_i, the relation (see, for instance, [5, p. 214])

$$\rho_i = \tan R_i \tag{2.2}$$

is well known.

Furthermore if $d\alpha_i$ is the angle between two infinitely near normals to S^{n-1} along C_i at their intersection point,

$$ds_i = \sin R_i d\alpha_i. \tag{2.3}$$

From (2.1) and (2.3) we deduce

$$dP = \prod_{i=1}^{n-1} \sin R_i d\alpha_i. \tag{2.4}$$

Applying (2.4) to the hypersurface $S^{n-1}(\lambda)$, we have

$$dP(\lambda) = \prod_{i=1}^{n-1} \sin (R_i + \lambda) d\alpha_i \tag{2.5}$$

or, according to (2.4),

$$
\begin{aligned}
dP(\lambda) &= \prod_{i=1}^{n-1} (\sin R_i \cos\lambda + \cos R_i \sin\lambda) d\alpha_i \\
&= \prod_{i=1}^{n-1} (\cos\lambda + \sin\lambda/\tan R_i) dP.
\end{aligned} \tag{2.6}
$$

From the definition (1.1) and from (2.2) we deduce

$$
M_i(\lambda) = \int_{S^{n-1}(\lambda)} \left(\sum \frac{1}{\tan(R_{r_1}+\lambda)\cdots\tan(R_{r_i}+\lambda)} \right) dP(\lambda), \tag{2.7}
$$

or, according to (2.5) and (2.4)

$$
\begin{aligned}
M_i(\lambda) = \int \sum \Bigg(& \prod_{j=1}^{i} \cos(R_{r_j}+\lambda) \\
& \cdot \prod_{j=i+1}^{n-1} \sin(R_{r_j}+\lambda) \Bigg) d\alpha_1 \cdots d\alpha_{n-1} \\
= \int_{E_n^{-1}} \sum \Bigg(& \prod_{j=1}^{i} \left(\frac{\cos\lambda}{\tan R_{r_j}} - \sin\lambda \right) \\
& \cdot \prod_{j=i+1}^{n-1} \left(\cos\lambda + \frac{\sin\lambda}{\tan R_{r_j}} \right) \Bigg) dP.
\end{aligned} \tag{2.8}
$$

The sums are always extended over all combinations of ith order of the indices $1, 2, \cdots, n-1$.

If we take into account (2.2) and the definition (1.1) of M_i, from the last equality results[2]

$$
M_i(\lambda) = \sum_{k=0}^{n-1} M_k \phi_{ik}(\lambda) \tag{2.9}
$$

where

$$
\phi_{ik}(\lambda) = \sum_{h=p}^{q} (-1)^{i-h} C_{n-1-k,i-h} C_{k,h} \sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda, \tag{2.10}
$$

where the sum is extended over all values of h for which the combina-

[2] The combinatory coefficients which appear in (2.10) are easily obtained if we observe that the number of terms in the sum (2.8) with k factors $1/\tan R_{r_j}$ and coefficient $\sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda$ is $C_{i,h}C_{n-1-i,k-h}C_{n-1,i}$ and the number of terms in the sum (1.1) which gives M_k is $C_{n-1,k}$. Therefore the product $M_k \sin^{i+k-2h}\lambda \cos^{n-1-i-k+2h}\lambda$ appears a number of times equal to the quotient of the two foregoing combinatory numbers, which is equal to $C_{n-1-k,i-h}C_{k,h}$.

tory symbols have a sense, that is

$$p = \max(0, i + k - n + 1), \qquad q = \min(i, k). \tag{2.11}$$

Formulas (2.9) and (2.10) solve our problem for the elliptic case.

3. **The hyperbolic case.** For the case of a hypersurface S^{n-1} in the hyperbolic n-dimensional space, formulas (2.2) and (2.3) must be replaced respectively by

$$\rho_i = \tanh R_i, \qquad ds_i = \sinh R_i d\alpha_i. \tag{3.1}$$

Exactly the same calculation as before gives now

$$M_i(\lambda) = \sum_{k=0}^{n-1} M_k \phi_{ik}(\lambda) \tag{3.2}$$

with

$$\phi_{ik}(\lambda) = \sum_{h=p}^{q} (-1)^{i-h} C_{n-1-k,i-h} C_{k,h} \sinh^{i+k-2h} \lambda \cosh^{n-1-i-k+2h} \lambda, \tag{3.3}$$

where p, q are given by (2.11).

4. **Polar surfaces.** In the elliptic case it is interesting to consider the polar surface $S^{n-1}(\pi/2)$ to the given S^{n-1}.

Applying (2.9), (2.10) for $\lambda = \pi/2$ we obtain

$$M_i(\pi/2) = (-1)^i M_{n-1-i}. \tag{4.1}$$

If M_i^P denotes the ith mean curvature of the polar surface, we have $M_i^P = (-1)^i M_i(\pi/2)$ and consequently

$$M_i^P = M_{n-1-i}. \tag{4.2}$$

For $i = 0$

$$A^P = M_{n-1},$$

which is a result due to Allendoerfer [1, formula (30)]. For $n = 3$ we get $A^P = M_2$, $M_1^P = M_1$ or, applying the Gauss-Bonnet formula (1.3)

$$M_1^P = M_1, \qquad A^P + A = -4\pi\chi'.$$

The last formula is due to Blaschke [4].

5. **Hypersurfaces of constant width.** Let us assume S^{n-1} to be a topological sphere such that the inward drawn normal at every point P cuts S^{n-1} beside P at only one opposite point P^*. Let Δ be the distance PP^* measured along the normal. If Δ is constant for every point

P, S^{n-1} is said to be a hypersurface of "constant width."

In such a case the normal at P^* coincides with P^*P. Indeed, if Q is a point of S^{n-1} such that the distance PQ is a maximum (P fixed, Q variable on S^{n-1}), QP must be normal to S^{n-1} at Q and therefore, by assumption, distance $QP=\Delta$; on the other hand, if P^*P were not normal to S^{n-1} at P^*, the distance PP^* would not be a maximum, thus distance $PP^*<$distance $PQ=\Delta$, contrary to the assumption.

Furthermore, according to the definition of the radii R_i and the assumption that they are not negative (see §1 and (2.2), (3.1)), the point of contact of the normal PP^* with the envelope of the normals along each line of curvature through P does lie inside the segment PP^*; therefore for the hypersurfaces of constant width, between the corresponding radii R_i, R_i^* at opposite points, the relation

$$R_i + R_i^* = \Delta, \qquad i = 1, 2, \cdots, n-1, \tag{5.1}$$

holds.

We have also $dP=(-1)^{n-1}dP^*$, and consequently (2.7) gives

$$M_i(-\Delta) = (-1)^{n-1-i}M_i \tag{5.2}$$

which holds the same in both elliptic and hyperbolic cases.

Therefore, taking into account the relations (2.9) and (3.2) we get:

Between the mean curvatures M_i of a hypersurface of constant width Δ in the elliptic or hyperbolic n-dimensional space, the relations

$$M_i = (-1)^{n-1-i}\sum_{k=0}^{n-1} M_k\phi_{ik}(-\Delta), \qquad i = 0, 1, 2, \cdots, n-1, \tag{5.3}$$

hold, where ϕ_{ik} are given by (2.10) *in the elliptic case and by* (3.3) *in the hyperbolic case.*

Furthermore, if V is the volume enclosed by S^{n-1}, we have $V(-\Delta)=(-1)^nV$ and therefore (1.2) and (2.9), (3.2) give the following relation

$$V = (-1)^nV + (-1)^n\sum_{k=0}^{n-1} M_k\int_0^{-\Delta}\phi_{0k}(\lambda)d\lambda, \tag{5.4}$$

which must be added to the preceding ones (5.3).

The obtained relations (5.3) and (5.4) are, in general, not independent, as the following examples will show.

EXAMPLE 1. If $n=2$, (5.3) and (5.4) are equivalent to the unique relation

$$M_0 \sin\Delta - M_1(1-\cos\Delta) = 0 \text{ (elliptic case)},$$

$$M_0 \sinh \Delta - M_1(1 - \cosh \Delta) = 0 \text{ (hyperbolic case).}$$

If L is the length and A the area enclosed by S^1, $M_0 = L$ and the Gauss-Bonnet formula gives $M_1 = 2\pi \pm A$; therefore the foregoing relations may be written respectively

$$(5.5) \qquad L = (2\pi - A) \tan (\Delta/2), \qquad L = (2\pi + A) \tanh (\Delta/2).$$

EXAMPLE 2. For $n = 3$, if we set $M_0 = A$ and take into account (1.3) which gives $M_2 = 4\pi \pm A$, the relations (5.3) become equivalent to

$$(5.6) \qquad \begin{aligned} M_1 \cos \Delta &= 2(2\pi - A) \sin \Delta \text{ (elliptic case),} \\ M_1 \cosh \Delta &= 2(2\pi + A) \sinh \Delta \text{ (hyperbolic case).} \end{aligned}$$

(5.4) gives

$$2V = 2\pi\Delta - (M_1/2) \sin^2 \Delta - (2\pi - A) \sin \Delta \cos \Delta \text{ (elliptic case),}$$

$$2V = -2\pi\Delta - (M_1/2) \sinh^2 \Delta + (2\pi + A) \sinh \Delta \cosh \Delta \text{ (hyperbolic case).}$$

If we take into account (5.6), the last relations can be written respectively

$$(5.7) \qquad 4V = 4\pi\Delta - M_1, \qquad 4V = M_1 - 4\pi\Delta.$$

(5.5) and (5.7) are due to Blaschke [3]. For the analogous questions in the n-dimensional euclidean space, see [7].

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THE NON-EUCLIDEAN MIRROR

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We propose to show in this paper that some laws of geometrical optics hold true for hyperbolic geometry. Likewise optical properties of certain curves can be carried over. All the proofs given will be valid for elliptic geometry, as long as there are no parallel lines involved.

Fermat's principle may be stated as follows: The path of a light ray from a point A to a point B under known conditions is such that the time taken by the light to traverse it is a minimum. In order to find the law of reflection based on this principle, we shall use Cartesian coordinates [6, p. 138].[1] Let $(0, a)$ be the coordinates of A and (c, b) those of B, where $a>0$, $b>0$. A and B are joined to a point $P(x, 0)$ on the x-axis (surface of the mirror). Then the function $z=AP+PB$ has to be minimized (cf. [2, p. 164]). By trigonometry [3, p. 238]: $\cosh AP=\cosh a \cosh x$, whence

$$\sinh AP \frac{d}{dx}(AP) = \cosh a \sinh x.$$

Finding the derivative of PB in the same way, we have

$$\frac{dz}{dx} = \frac{\cosh a \sinh x}{\sinh AP} - \frac{\cosh b \sinh (c-x)}{\sinh PB} = 0.$$

If we now denote by α and β the acute angles formed by AP and PB with the x-axis, we obtain the trigonometric relations

$$\cos \alpha = \frac{\tanh x}{\tanh AP} = \frac{\cosh a \sinh x}{\sinh AP}, \qquad \cos \beta = \frac{\cosh b \sinh (c-x)}{\sinh PB}.$$

Thus,

$$\cos \alpha = \cos \beta \quad \text{or} \quad \alpha = \beta,$$

that is, the angle of incidence is equal to the angle of reflection for the required minimum. The second derivative of z is actually positive for any x. The same result could be found by a simple geometric method; but in its analytical form the above procedure can easily be generalized to yield the law of refraction.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Now suppose we want to solve the following problem: Find the shape of a mirror such that all light coming from one fixed point is reflected to another fixed point [5, p. 29]. A synthetic solution of this problem in Euclidean geometry, which applies to the hyperbolic case without change, can be found in R. Courant and H. Robbins, *What is mathematics?*, London, 1941, p. 333. Let the two fixed points be $F_1(-c, 0)$ and $F_2(c, 0)$. Take a characteristic point $P(x, y)$ of the curve, $Q(x, 0)$ being its projection on the x-axis. Let $PF_1=r_1$, $PF_2=r_2$, $\angle F_1PQ=\theta$, and $\angle F_2PQ=\phi$. Designate by α the angle through which PQ must be rotated in the counterclockwise sense in order to coincide with the tangent line at P. Then, according to the above law of reflection:

$$\alpha - \phi = \pi - \alpha - \theta, \quad \text{and} \quad \cos(\alpha - \phi) + \cos(\alpha + \theta) = 0,$$

or

$$\cos\theta \cot\alpha - \sin\theta + \cos\phi \cot\alpha + \sin\phi = 0. \tag{1}$$

It has been shown [4] that

$$\cot\alpha = -\operatorname{sech} y \frac{dy}{dx}.$$

On the other hand, from the right triangle F_1PQ:

$$\cos\theta = \frac{\tanh y}{\tanh r_1} = \frac{\cosh(x+c)\sinh y}{\sinh r_1}, \qquad \sin\theta = \frac{\sinh(x+c)}{\sinh r_1}.$$

Using the analogous relations for ϕ and substituting all this, we see that (1) becomes

$$\frac{\cosh(x+c)\tanh y dy + \sinh(x+c)dx}{\sinh r_1} + \frac{\cosh(x-c)\tanh y dy + \sinh(x-c)dx}{\sinh r_2} = 0. \tag{2}$$

Again, $\cosh r_1 = \cosh(x+c)\cosh y$, and by differentiation

$$\sinh r_1 dr_1 = \cosh(x+c)\sinh y dy + \sinh(x+c)\cosh y dx.$$

Thus, (2) reduces to

$$dr_1 + dr_2 = 0.$$

Hence

$$r_1 + r_2 = \text{const.}.$$

so that the only curves having the required property are ellipses [1, p. 146 and 148].

Let us consider one more problem: Find the surface of a mirror such that all light from a fixed point is reflected parallel to a fixed line. Let the fixed point coincide with the origin and take P and Q as before. Let $OP=r$ and $\angle OPQ=\theta$. The angle α has the same meaning as before and $\phi=\Pi(y)$ denotes the angle of parallelism for y. If we take the directed x-axis as the fixed line of our problem, we have from the law of reflection the above relation (1), but now

$$\cos\theta=\frac{\cosh x\sinh y}{\sinh r},\qquad \sin\theta=\frac{\sinh x}{\sinh r}.$$

Also [6, p. 151],

$$\cos\phi=\tanh y,\qquad \sin\phi=\operatorname{sech} y.$$

Hence, (1) becomes upon substitution

$$(3)\qquad \frac{\cosh x\tanh y\,dy+\sinh x\,dx}{\sinh r}+\tanh y\operatorname{sech} y\,dy-\operatorname{sech} y\,dx=0.$$

As is seen in [6, p. 165], the distance from P to the limiting curve through O, which is perpendicular to all the parallels to the x-axis, is represented by

$$\xi=x-\log\cosh y.$$

Thus, $d\xi=dx-\tanh y\,dy$, and (3) is reduced to

$$dr-d\xi=0.$$

Hence

$$r-\xi=\text{const.},$$

which represents a family of parabolas.

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A THEOREM ON POLYNOMIAL IDENTITIES

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1. **Introduction.** In a recent paper Kaplansky [4][1] has emphasized the importance of rings which satisfy a polynomial identity (in short: PI-rings) and proved a number of interesting theorems concerning such rings. In the present note some further properties of PI-rings are derived. In the following, a PI-ring is said to be of degree d if d is the degree of a "minimal equation" (that is, a polynomial identity of minimal degree). In ascribing the term radial to the sum of all nilpotent ideals (notation: $N(S)$ = radical of S) we may express our main result as follows: The nilpotent elements of a PI-ring[2] are of bounded index modulo the radical. More precisely: The nilpotent elements of the quotient-ring $S/N(S)$ satisfy the identity $x^{[d/2]}=0$. This is proved in §3 (Theorem 1), where also some immediate consequences of this theorem are listed. Extensive use is made of Kaplansky's Lemma 2 [4] which is restated in §2 of the present paper (Lemma 3) in a slightly generalized form.

In §4 we discuss in some detail an application to PI-nil-rings (in short: NPI-rings). Our result in this section is connected with the following construction due to Baer [1]: By means of the transfinite induction he defines the rth radical $N_r(S)$ as follows: (1) $N_1(S)=N(S)$. (2) If $r=t+1$, then $N_r(S)$ is uniquely determined by the relations $N_r(S)\supseteq N_t(S)$, $N_r(S)/N_t(S)=N[S/N_t(S)]$. (3) If r is a limit-ordinal, then $N_r(S)=\bigcup_{t<r}N_t(S)$. There exists a smallest ordinal λ such that $N_\lambda(S)=N_{\lambda+1}(S)$. The ideal $N_\lambda(S)$ will be called the ultimate radical[3] of S and denoted by $U(S)$. For this ideal $N[S/U(S)]=0$, and if for an ideal A the relation $N(S/A)=0$ holds, then $A\supseteq U(S)$. Since also the ideal $N^*(S)$, defined as the sum of all semi-nilpotent ideals, has the property $N[S/N^*(S)]=0$ (see [5]), it follows that $N^*(S)\supseteq U(S)$, that is, the ultimate radical is semi-nilpotent.

A ring T which coincides with its ultimate radical will be called an L-ring. The smallest ordinal λ for which $N_\lambda(T)=N_{\lambda+1}(T)$ will be called the length of T (notation: $\lambda=\lambda(T)$ = length of T). It is an immediate consequence of the definition that each L-ring is semi-nilpotent, and hence a nil-ring.

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] As to the domain of coefficients of the polynomial identities, compare §2.

[3] This ideal was termed by Baer the lower radical. For the purpose of the present note the term ultimate radical seems the more appropriate.

It has been proved by Kaplansky [4] that an NPI-ring is semi-nilpotent. In section 4 it is shown that an NPI-ring is an L-ring. It should be pointed out in this connection that whereas it is unknown whether or not nil-rings exist which are not semi-nilpotent, it can be deduced from an example due to Baer (see [1, §2]) that a semi-nilpotent ring need not be an L-ring.[4] We further prove that the length $\lambda(S)$ of an NPI-ring S is finite, and that the degree $d(S)$ is an upper bound for $\lambda(S)$. More precisely: If $S\neq 0$, then $\lambda(S) \leqq \log d(S)/\log 2$. (If $S=0$ then $\lambda=d=1$.) An immediate consequence of Theorem 1 is: If S is an NPI-ring, then the quotient-ring $S/N(S)$ is of bounded index. Moreover, the identity $x^{[d/2]}=0$ holds in $S/N(S)$.

2. **Preliminary remarks.** In the case of algebras the coefficients of the polynomial identities are chosen from the underlying field. In order to include this case in our deliberations, we assume that the coefficients $\alpha, \beta, \gamma, \cdots$ of our polynomials belong to a domain D of operators having the following properties:

I. The domain D is a subset of the ring E of endomorphisms of the additive group defined by the given ring S.

II. For $\alpha\in D$; $s_1, s_2\in S$, we have $\alpha(s_1s_2)=(\alpha s_1)s_2=s_1(\alpha s_2)$.

III. If $\alpha\in D$, then either $\alpha S=0$ (that is, α is the null-endomorphism) or α is an automorphism. In the latter case we assume that also α^{-1} belongs to D.

IV. The domain contains the endomorphisms $0, \pm 1$, and if $\alpha\in D$, then also $-\alpha\in D$.

Each ring possesses an operator domain with properties I–IV, for example, the set of the 3 numbers $0, \pm 1$.

A sub-ring A of S is called D-admissible, if for each α we have $\alpha S\subseteq S$.

LEMMA 1. *If $a\in S$, then Sa, aS, and SaS are D-admissible.*

PROOF. For $\alpha(Sa)=(\alpha S)a\subseteq Sa$; $\alpha(aS)=a(\alpha S)\subseteq aS$; $\alpha(SaS)=(\alpha S)aS \subseteq SaS$.

LEMMA 2. *If A is a D-admissible sub-ring of S and $0\neq\alpha\in D$, then $\alpha A=A$.*

PROOF. For we have $\alpha A\subseteq A=\alpha(\alpha^{-1}A)\subseteq\alpha A$.

The polynomials which are considered in this paper are elements of the free algebra $E(x_1, x_2, \cdots, x_n)$ generated (compare [4]) by the

[4] This shows incidentally that the ring constructed by Baer furnishes an example of a nil-ring for which no polynomial identity can be found.

indeterminates $x_1, \cdots, x_n$ over the ring of endomorphisms E, but we restrict our attention only to such polynomials whose coefficients belong to D and whose constant coefficient is 0. If the polynomial identity

$$f(x_1, x_2, \cdots, x_n) = 0 \tag{1}$$

is not linear in one of the indeterminates, say x_1, one performs (see [4, Lemma 2]) the transformation $g(u, t, x_2, \cdots, x_n) = f(u+t, x_2, \cdots, x_n) - f(u, x_2, \cdots, x_n) - f(t, x_2, \cdots, x_n)$. One obtains the identity $g(u, t, x_2, \cdots, x_n) = 0$ which has a lower degree in u as well as in t than the degree of (1) in x_1. The general degree of g is not higher than that of f, and if the coefficients of f belong to D, then this holds also for the coefficients of g. If, finally, an indeterminate, say x_1, does not appear in at least one monomial with a nonzero coefficient, and if $h(x_2, \cdots, x_n)$ is the sum of all such monomials, then evidently $h(x_2, \cdots, x_n) = 0$ holds in S. As a consequence of these remarks we have the following lemma.

LEMMA 3. *If S is a PI-ring of degree d, then S satisfies a polynomial identity of the form*

$$f(x_1, x_2, \cdots, x_d) = \sum_{(i)\in\Pi} \alpha_{(i)} x_{i_1} \cdots x_{i_d} = 0 \tag{2}$$

where Π is a set of permutations (i) of the d numbers $1, 2, \cdots, d$ and $\alpha_{(i)} \in D$.

3. **The main theorem.** We assume in this section that the ring S has an operator domain D satisfying conditions I–IV of §2 and that the coefficients of all polynomial identities belong to this domain.

THEOREM 1. *If S is a PI-ring of degree d and $N(S)$ the radical of S, then for each nilpotent element a of S we have*

$$a^{[d/2]} \in N(S). \tag{3}$$

PROOF. This is true for $a \in N(S)$. Now suppose that $a \notin N(S)$, and denote by n the index of a modulo $N(S)$, that is,

$$a^n \notin N(S), \qquad a^{n+1} \in N(S). \tag{4}$$

The theorem will be proved if we show that $n < [d/2]$. To this end consider the following $2n+1$ sub-rings of S

$$\begin{aligned} A_{2j-1} &= a^{n-j+1} S a^{j-1}, \qquad j = 1, 2, \cdots, n+1, \\ A_{2j} &= a^{n-j+1} S a^{j}, \qquad\quad j = 1, 2, \cdots, n, \end{aligned} \tag{5}$$

and form the products

$$B_i = A_1 \cdot A_2 \cdots A_i, \qquad i = 1, 2, \cdots, 2n+1. \tag{6}$$

It follows that

$$\begin{aligned} B_{2j-1} &= (a^nS)^{2j-1}a^{j-1}, \quad j = 1, 2, \cdots, n+1, \\ B_{2j} &= (a^nS)^{2j}a^j, \qquad\qquad j = 1, 2, \cdots, n. \end{aligned} \tag{7}$$

From (5) we deduce that $A_sA_t \subseteq Sa^{n+1}S$ for $s>t$. Hence if for any integer r such that $r \leqq 2n+1$ one denotes by $(i_1, \cdots, i_r)$ a permutation of the integers $1, 2, \cdots, r$, we have

$$A_{i_1}A_{i_2} \cdots A_{i_r} \subseteq Sa^{n+1}S \qquad \text{if } (i_1 \cdots i_r) \neq (1, \cdots, r) \tag{8}$$

or in other words

$$\begin{aligned} &a_{i_1}a_{i_2} \cdots a_{i_r} \in Sa^{n+1}S \\ &\text{for } a_k \in A_k,\ k = 1, \cdots, r \text{ and } (i_1, \cdots, i_r) \neq (1, \cdots, r). \end{aligned} \tag{9}$$

We turn now to the given polynomial identity which according to Lemma 3 we may write in the form

$$\beta x_1x_2 \cdots x_d = \sum_{(i)\in\Pi'} \beta_{(i)}x_{i_1} \cdots x_{i_d}, \qquad \beta \neq 0, \tag{10}$$

where the permutations (i) of the set Π' are all different from the identical permutation. Suppose now that $n \geqq [d/2]$; then we have $n \geqq (d-1/2)$ or $d \leqq 2n+1$. Hence we may substitute in (9) the integer d for r, thus obtaining

$$a_{i_1}a_{i_2} \cdots a_{i_d} \in Sa^{n+1}S \qquad \text{for } (i_1 \cdots i_d) \neq (1, \cdots, d). \tag{11}$$

Since $\beta_{(i)}Sa^{n+1}S \subseteq Sa^{n+1}S$ (Lemma 1) we obtain in view of[5] (10)

$$\beta a_1a_2 \cdots a_d \in Sa^{n+1}S \tag{12}$$

or $\beta A_1 \cdots A_d = \beta B \subseteq Sa^{n+1}S$. By formulas (7) we now have

$$\begin{aligned} \beta B_d &= \beta(a^nS)^{2q-1}a^{q-1} \subseteq Sa^{n+1}S, \qquad \text{if } d = 2q-1, \\ \beta B_d &= \beta(a^nS)^{2q}a^q \subseteq Sa^{n+1}S \qquad\quad \text{if } d = 2q. \end{aligned} \tag{13}$$

Right multiplication of the first relation in (13) by $a^{n-q+1}Sa^nS$, and

[5] A slight simplification of the argument is possible by operating in the quotient-ring $S/Sa^{n+1}S$. We have then zero on the right side of formulas (8), (9), and (11)–(14). Passage to this quotient-ring is permissible since the ideal $Sa^{n+1}S$ is admissible. This ensures the survival of the polynomial identity and, in view of postulate III, also of its degree; each nonzero endomorphism of S becomes a nonzero endomorphism of $S/Sa^{n+1}S$.

of the second by $a^{n-q}S$, yields in both cases in view of $\beta(a^nS)=a^nS$ (Lemmas 1 and 2)

$$(a^nS)^{2q+1} \subseteq Sa^{n+1}S. \tag{14}$$

Now we assumed in (4) that $a^{n+1}\in N(S)$, which implies that the ideal $Sa^{n+1}S$ is nilpotent. For some integer t we have therefore $(a^nS)^{t(2q+1)}=0$, which implies that $a^n\in N(S)$, contradicting (4). We were led to this contradiction by assuming that $n\geqq[d/2]$. Hence $n<[d/2]$, q.e.d.

Consequences. (1) Let S denote a PI-ring of degree d, and T a D-homomorphic image of S. If $N(T)=0$, then the nilpotent elements of T are of bounded index. Moreover, *these elements satisfy the identity* $x^{[d/2]}=0$. Indeed, the polynomial identity satisfied by S as well as its degree are inherited by T, and our assertion is therefore a direct consequence of Theorem 1.

(2) If S is the ring of n by n matrices over a commutative or a noncommutative field F, and if d is the degree of a polynomial identity satisfied by S, then $n\leqq[d/2]$. Indeed, this is a consequence of $N(S)=0$ and of the fact that S contains nilpotent elements of index $n-1$. This implies Kaplansky's Lemma 5 [4] which he applies in the proof of his Theorem 1 which states that each PI-primitive algebra in the sense of Jacobson [3] is finite-dimensional over its center. As to the problem raised by Kaplansky concerning an explicit upper bound for the order of the algebra over its center, our exponent $[d/2]$ yields a rather sharp estimate. By using Kaplansky's Lemma 3 [4] we have the following consequence.

(3) If k^2 is the order of a simple algebra over its center, and d is the degree of a polynomial identity, then we have

$$k^2 \leqq [d/2]^2. \tag{15}$$

(4) In case $d=1$ the situation is trivial ($S=0$). Also the case $d=2$, which is a slight generalization of the commutative case, yields nothing new. We have here $[d/2]=1$, that is, all nilpotent elements belong to $N(S)$, and if S is an algebra of finite order, then $S/N(S)$ is in view of (3) a direct sum of commutative fields.

(5) In case $d=3$ we have $[d/2]=1$, that is, again all nilpotent elements of S belong to $N(S)$. It is of interest to note that just as in the commutative case we have also here the coincidence of all radical-like ideals which have been hitherto defined by using various types of nillity. If S is an algebra of finite order it follows in view of (3) that $S/N(S)$ is the direct sum of commutative fields, a fact which was also mentioned by Kaplansky.

(6) In case $d=4$ or $d=5$ we have $[d/2]=2$. If S is a division algebra which satisfies a polynomial identity of degree 4 or 5, it follows in view of (3) that S is either a commutative field or a generalized quaternion algebra. This was previously proved by Hall [2, Theorem 6.2] for the case of a specific polynomial identity of degree 5. Compare also [4].

(7) A direct simple proof of statement (2) may be deduced as follows: Consider the n^2 matrices c_{ij}, $i, j=1, \cdots, n$, satisfying the relations

$$c_{ij}c_{jk} = \begin{cases} 0 & \text{if } j \neq j', \\ c_{ik} & \text{if } j = j'. \end{cases}$$

Put $a_{2i-1}=c_{ii}$, $a_{2i}=c_{ii+1}$ and form the product $a_1a_2 \cdots a_r$ for each positive integer $r<2n$. Suppose that $d(S)<2n$ and write the polynomial identity in the form (10). By substituting a_i for x_i, $i=1, \cdots, d$, one obtains zero on the right side of (10) and something not equal to 0 on the left. This contradiction is a consequence of $d<2n$. Hence $d \geqq 2n$, q.e.d.

4. **Polynomial identities and L-rings.** Also in this section we assume that all polynomial identities considered have their coefficients in a domain D as defined in §2.

THEOREM 2. *If S is an NPI-ring of degree d, and $N_r(S)$ is the rth radical of S, then the ring $S/N_r(S)$ satisfies the identity*

$$x^{[d/2]} = 0. \tag{16}$$

PROOF. For $r=1$ this is an immediate consequence of Theorem 1. For $r>1$ this follows from $N_r(S) \supseteq N_1(S)$.

THEOREM 3. *If S is an NPI-ring and $S \supset 0$, then also $N(S) \supset 0$.*[6]

PROOF. If d is the degree of S, then for each $x \in S$ we have by Theorem 1 the relation $x^{[d/2]} \in N(S)$. From $S \neq 0$ it follows that $d \geqq 2$. In case $2 \leqq d \leqq 3$, we have $[d/2]=1$, that is, $x \in N(S)$ or $S=N(S)$, and the theorem is proved. We consider now the case $d \geqq 4$ and suppose that $N(S)=0$. It follows then that the identity $x^{[d/2]}=0$ whose degree is smaller than d holds in S, which is a contradiction. Hence $N(S) \neq 0$, q.e.d.

THEOREM 4. *Each NPI-ring is an L-ring.*

PROOF. By Theorem 2 the quotient-ring $S/N(S)$ satisfies an iden-

[6] This theorem is in some respect a generalization of the fact that for each commutative nil-ring we have $N(S)=S$.

tity of the form $x^p=0$. Since $U(S)\supseteq N(S)$, this identity is satisfied also in $S/U(S)$. In view of $U[S/U(S)]=0$ we have also $N[S/U(S)]=0$ and hence it follows by Theorem 3 that $S/U(S)=0$, or $S=U(S)$, q.e.d.

COROLLARY. *Denote by S an arbitrary PI-ring. If $U(S)$ is the ultimate radical and $U'(S)$ is the upper radical (see Baer* [1]*), then*: (1) *$U(S)=U'(S)$ and* (2) *the quotient-ring $S/U(S)$ has no one-sided nil-ideals other than zero.*

Indeed, since $U(S)$ is D-admissible (see [1]) the polynomial identity, and in view of postulate III in §2 also its degree are inherited by $U'(S)/U(S)$. Since $N[U'(S)/U(S)]=0$ (compare [1]) it follows by Theorem 3 that $U'(S)/U(S)=0$, or $U'(S)=U(S)$. To prove the second part of the above statement, suppose that R is a right (left) nil-ideal in S. By Theorem 4 it follows that R is an L-ring, and hence semi-nilpotent. We have therefore $R\subseteq U^*(S)$, where $U^*(S)$ is the sum of all semi-nilpotent ideals of S. Since $U^*(S)\subseteq U'(S)=U(S)$, we have $R\subseteq U(S)$, which implies the second part of our statement.

THEOREM 5. *Let S be an NPI-ring of degree d_0, $N_r(S)$ the rth radical of S, and d_r the degree of the quotient-ring $S_r=S/N_r(S)$. Then*:

(1) *If $d_r>1$ then $d_{r+1}\leqq d_r/2$.*

(2) *The ring S, which by Theorem 4 is an L-ring, has finite length.*

(3) *If $\lambda=\lambda(S)$ denotes the length of S, then*

$$\lambda \leqq \log d_0/\log 2 \text{ for } d_0>1. \text{ If } d_0=1, \text{ then } \lambda=1. \tag{17}$$

PROOF. (1) For $r=0$ this is an immediate consequence of Theorem 1. Now assume $r>0$. We know (for example, by Theorem 2) that S_r satisfies a polynomial identity. If d_r is the degree of S_r, then by Theorem 2 the ring $S_r/N(S_r)$ satisfies the identity

$$x^{[d_r/2]}=0. \tag{18}$$

By definition $N(S_r)=N_{r+1}(S)/N_r(S)$ and therefore $S_r/N(S_r)=[S/N_r(S)]/[N_{r+1}(S)/N_r(S)]\cong S/N_{r+1}(S)=S_{r+1}$. This isomorphism implies that identity (18) is satisfied by S_{r+1}. If now d_{r+1} is the degree of S_{r+1}, we have therefore $d_{r+1}\leqq[d_r/2]\leqq d_r/2$, q.e.d.

(2) Suppose that $d_r>1$ for each finite r; then by (1) we would obtain an infinite sequence $d_1>d_2>\cdots$, a contradiction. Hence there exists a finite index j so that $d_j=1$. This means that the ring $S/N_j(S)$ satisfies an identity of degree 1, and hence $S/N_j(S)=0$, or $S=N_j(S)$. By the definition of length we have therefore $\lambda\leqq j$, q.e.d.

(3) If $d_0=1$, then $0=S=N_1(S)$, or $\lambda=1$ in which case (3) is proved. If $d_0>1$ and $\lambda=1$, then (3) is also true. Now suppose that $\lambda>1$, then evidently $d_r>1$ for $r\leqq\lambda-1$ (otherwise we would have $S=N_r(S)$ for some r which is smaller than λ, contradicting the definition of length). This implies in view of (1) that $d_\lambda\leqq d_0/2^\lambda$ or $2^\lambda\leqq d_0/d_\lambda$, that is, $\lambda \log 2\leqq\log d_0-\log d_\lambda\leqq\log d_0$, q.e.d.

COROLLARY. *The length of an NPI-ring is smaller than the degree of the identity. Equality holds only in the trivial case* $S=0$.

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HEBREW UNIVERSITY

A REMARK ON A THEOREM OF MARSHALL HALL

M. F. SMILEY

Marshall Hall [2][1] proved that the identity

$$x(yz - zy)^2 = (yz - zy)^2x \tag{1}$$

characterizes quaternion algebras among associative but not commutative division rings. Our remark is that (1) characterizes Cayley-Dickson algebras among alternative but not associative division rings. This follows from Hall's proof and a result of A. A. Albert [1]. A theorem of R. D. Schafer [4, Theorem 4] permits us to conclude that (1) and Hall's Theorem L [2] (as universal) ensure that the coordinate ring of a projective plane is uniquely defined irrespective of the coordinate system.

It is easy to verify (1) in a Cayley-Dickson algebra. We should like to sketch a proof of the converse, which is independent of Albert's result in [1]. Let A be an alternative division ring and let F be the set of all elements $c \in A$ which satisfy $cx = xc$ for every $x \in A$. Then F is a field[2] and, when 3 is nonzero in A, F is the center of A [3; 5, Lemma 9]. If A satisfies (1), the proof of Hall's Lemma 1 [2, p. 262] yields $x^2 = t(x)x - n(x)$ for every $x \in A$ not in F, where $t(x), n(x) \in F$. Define $t(c) = 2c$, $n(c) = c^2$ for $c \in F$.

When 3 is zero in A and A satisfies (1), F is still the center of A as we shall now show.[3] Since 2 is a nonzero in A, for each $x \in A$ we have $(x+c)^2 \in F$ for some $c \in F$. If, then, c_1, c_2 are in F, we have $(c_1, c_2, (x+c)^2) = 0$. By use of some identities of Zorn [6, (1.6) and (1.8); 5, (1)] we find that $(c_1, c_2, (x+c)^2) = 2(x+c)(c_1, c_2, x+c) = 2(x+c) \cdot (c_1, c_2, x)$. We infer that $(c_1, c_2, x) = 0$ for every c_1, c_2 in F and every x in A. Now let $c \in F$ and x, $y \in A$. As before, $(x+c')^2 \in F$ for some $c' \in F$ and

$$0 = (c, (x + c')^2, y) = 2(x + c')(c, x + c', y) = 2(x + c')(c, x, y),$$

so that $(c, x, y) = 0$ as desired.

To complete the argument, we first prove (i) $t(x+y) = t(x) + t(y)$,

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] To verify that $c, c' \in F$ yields $cc' \in F$ even if $3 = 0$ in F, we may use the equations $(x, c, c') = (c', x, c) = (c, c', x)$, where $(x, y, z) = x(yz) - (xy)z$ is the *associator* of x, y, and z.

[3] We are indebted to the referee for his observation that our original proof was incomplete in this case and also for simplifying our amended version.

(ii) $t(cx)=ct(x)$, and (iii) $n(xy)=n(x)n(y)$ for every x, $y\in A$ and every $c\in F$, and then we apply the arguments of R. D. Schafer [4]. We indicate the proof of (i) and (iii) when $xy\neq yx$. We compute $(x+y)^2=x^2+xy+yx+y^2=t(x+y)(x+y)-n(x+y)$, and we obtain (i) by taking the commutator with x and then (iii) by multiplying by xy on the left.

Addendum (October 25, 1949). When $1+1=0$ in A, we cannot obtain our result from Albert's Theorem 1 of [7] which employs $1+1\neq 0$. In this case we must complete the proof of (i)–(iii) by using Lemma 2 of [2, p. 262] and some straightforward but rather lengthy calculations. In fact, we may regard our proof as establishing Albert's Theorem 1 even when $1+1=0$. First redefine *quadratic algebra* by deleting the coefficient 2 in $f(\xi, x)$. Then assume that A is quadratic and alternative and observe that A is a division algebra. Unless A is commutative and hence a field over F, one may use the arguments of Hall [2, pp. 262–263] to see that the center of A is F. For, if $xy\neq yx$, then $Q=F[1, x, y, xy]$ is a quaternion algebra over F, and the requirement that $c\in A$ commute and associate with the elements of Q implies that $c\in Q$ and hence that $c\in F$, since the center of Q is F. We are now in a position to apply the arguments of the present note.

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State University of Iowa

A THEOREM ON THE EXTENSION OF A RECTANGULAR MATRIX OF CONTINUOUS FUNCTIONS TO BECOME A NONSINGULAR SQUARE MATRIX

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THEOREM. *If the elements* $a_{ij}(t)$ $(i=1, \cdots, p<q;\ j=1, \cdots, q)$ *of a* $p \times q$ *rectangular matrix* A *are continuous functions of* n *variables*[1] $t_1, \cdots, t_n$ *on a closed region* R *which is homeomorphic to a closed* n*-cell and* A *has maximum rank everywhere on* R*, then there exist polynomials* $b_{kj}(t)$ $(k=p+1, \cdots, q)$ *in* t *such that the matrix*

$$M = \begin{pmatrix} a_{ij}(t) \\ b_{kj}(t) \end{pmatrix}$$

is nonsingular everywhere on the closed region R.

There exists a homeomorphic mapping $u=\phi(t)$ $(u=u_1, \cdots, u_n)$ of R onto a closed n-dimensional cube K with one vertex at the origin and with the n coordinate axes as edges. The matrix $B=(a_{ij}(\phi^{-1}(u)) =\alpha_{ij}(u))$ has maximum rank everywhere on K and its elements are continuous on K. If the functions $\beta_1(u), \cdots, \beta_q(u)$ are continuous solutions of the equations

$$\sum_{j=1}^{q} \alpha_{ij}(u)\beta_j(u) = 0 \qquad (i = 1, \cdots, p) \tag{1}$$

which do not vanish simultaneously on K, the matrix

$$C = \begin{pmatrix} \alpha_{ij}(u) \\ \beta_j(u) \end{pmatrix}$$

has maximum rank everywhere on K.

At any point $\bar{u}$ on K at least one of the $p \times p$ determinants of the matrix B is different from zero in a neighborhood of $\bar{u}$. Consequently the cube K can be divided into a finite number of cubes k_s, the length of whose edges are equal and whose edges are parallel to the n coordinate axes, so that for each cube k_s at least one $p \times p$ determinant of the matrix B does not vanish on the closed cube k_s. If the proper $q-p$ β's are arbitrarily chosen on one of the cubes k_s so that they are continuous, the remaining β's will be uniquely determined and con-

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[1] For the special case where $n=1$, see G. A. Bliss *The problem of Mayer with variable end-points*, Trans. Amer. Math. Soc. vol. 19 (1918) p. 312.

tinuous on k_i. At any point $\bar{u}$ on the cube K, more than one $p \times p$ determinant may be different from zero and corresponding to each of these determinants there is a different set of arbitrary β's. If any one of these sets of arbitrary β's vanish simultaneously at $\bar{u}$, all of the β's must vanish at $\bar{u}$.

Starting with the small cube which has a vertex at the origin, denote this closed cube by k_1 and choose the arbitrary β's for this cube so that they are continuous and do not vanish simultaneously on k_1. Next choose any cube k_2 which has a common face with k_1. By the preceding argument, the arbitrary β's for k_2 can be extended as constants along straight lines through the origin. When the arbitrary β's for k_2 have been extended in this manner, the β's are uniquely determined, continuous, and do not vanish simultaneously on k_1+k_2. This process can be continued by choosing the cubes in such an order that the distance of k_e from the origin is equal to or less than that for k_f where $e<f$. Then the arbitrary β's for any cube can be extended as straight lines through the origin. Thus it is possible to find solutions of (1) which do not vanish simultaneously on K.

Since the matrix C has maximum rank everywhere on K, this process can be continued until a $q \times q$ matrix

$$D(u) = \begin{pmatrix} \alpha_{ij}(u) \\ \beta_{kj}(u) \end{pmatrix}$$

is obtained which is nonsingular everywhere on K. The matrix

$$D(\phi(t)) = \begin{pmatrix} a_{ij}(t) \\ \beta_{kj}(\phi(t)) \end{pmatrix}$$

is nonsingular everywhere on R and the functions $\beta_{kj}(\phi(t))$ are continuous on R. The functions $\beta_{kj}(\phi(t))$ can be approximated by polynomials $b_{kj}(t)$ so that the matrix M is nonsingular everywhere on R.

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ON THE NUMBER OF PRIMES LESS THAN OR EQUAL x

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In the beginnings of the theory of the distribution of prime numbers the unique factorization of $x!$ is found to be

$$x! = \prod_{p \leq x} p^{\alpha_p} \tag{1}$$

where

$$\alpha_p = \left[\frac{x}{p}\right] + \left[\frac{x}{p^2}\right] + \cdots .$$

Taking the logarithm of both sides of (1) and using the very elementary result

$$\log x! = x \log x + O(x) \tag{2}$$

there results (see [1, pp. 72, 98])[1]

$$\sum_{p \leq x} \left[\frac{x}{p}\right] \log p = x \log x + O(x). \tag{3}$$

Since $[x/p] = [[x]/p]$, it is clear that (3) then holds for all real $x > 0$.

We propose to show in this note that the order of $\pi(x)$ = the number of primes less than or equal x (a result originally due to Tschebyschef, cf. [1]) may be derived very quickly from (3) as a consequence of a general theorem which has no particular relationship to prime numbers. This is of some further interest in that Landau, in [1, p. 82], says effectively that (3) does not suffice to give Tschebyschef's theorem. Our general theorem, which is in the nature of a weak Tauberian theorem, is as follows.

THEOREM. *Let $a_n \geqq 0$ be a sequence of real numbers such that*

$$A(x) = \sum_{n \leq x} \left[\frac{x}{n}\right] a_n = x \log x + O(x). \tag{4}$$

Then for $S(x) = \sum_{n \leq x} a_n$ there exist two positive constants α, β, such that for all sufficiently large x

$$\beta x \geqq S(x) \geqq \alpha x. \tag{5}$$

PROOF. Since $[z] - 2[z/2] \geqq 0$ for all positive z, we have from (4)

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$$O(x) = A(x) - 2A\left(\frac{x}{2}\right) = \sum_{n \leqq x} \left\{\left[\frac{x}{n}\right] - 2\left[\frac{x}{2n}\right]\right\} a_n$$

$$\geqq \sum_{x/2 < n \leqq x} \left[\frac{x}{n}\right] a_n = \sum_{x/2 < n \leqq x} a_n = S(x) - S\left(\frac{x}{2}\right).$$

Thus for some positive K_1 we have $0 \leqq S(x) - S(x/2) < K_1 x$. Then $0 \leqq S(x/2^i) - S(x/2^{i+1}) < K_1 x/2^i$, from which we obtain via addition $S(x) < 2K_1 x$. Taking $\beta = 2K_1$ yields the upper bound in (5).

Having proved that $S(x) = O(x)$ we return to (4) and obtain

$$\sum_{n \leqq x} \left[\frac{x}{n}\right] a_n = \sum_{n \leqq x} a_n \left\{\frac{x}{n} + O(1)\right\} = x \sum_{n \leqq x} \frac{a_n}{n} + O(S(x))$$

$$= x \log x + O(x)$$

whence

$$T(x) = \sum_{n \leqq x} \frac{a_n}{n} = \log x + O(1). \tag{6}$$

Thus we can write $T(x) = \log x + R(x)$ where $|R(x)| \leqq K_2$ for all sufficiently large x. Choosing $\alpha = e^{-2K_2 - 1}$, we have

$$T(x) - T(\alpha x) = \sum_{\alpha x < n \leqq x} \frac{a_n}{n} = \log \frac{1}{\alpha} + R(x) - R(\alpha x).$$

From this we obtain

$$\frac{1}{\alpha x} S(x) \geqq \frac{1}{\alpha x} \sum_{\alpha x < n \leqq x} a_n \geqq \log \frac{1}{\alpha} - 2K_2 = 1$$

or

$$S(x) \geqq \alpha x,$$

as desired. This completes the proof of the theorem.

In order to apply the above theorem to (3) we take

$$a_n = \begin{cases} \log n & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

so that (4) reduces to (3). Then the theorem gives that for $\theta(x) = \sum_{p \leqq x} \log p$, we have two positive constants α, β, such that

$$\beta x \geqq \theta(x) \geqq \alpha x.$$

From the obvious inequalities

$$\{\pi(x) - \pi(x^{1/2})\} \log x^{1/2} \leqq \theta(x) \leqq \pi(x) \log x,$$

Tschebyschef's theorem

$$\alpha \frac{x}{\log x} \leqq \pi(x) \leqq \gamma \frac{x}{\log x}$$

now follows at once.

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THE RADICAL OF A NON-ASSOCIATIVE RING

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In this paper a definition is proposed for the radical of a non-associative ring. Our results are somewhat similar to those given for algebras by Albert in [3],[1] but the difficulties that arose in the earlier theory from absolute divisors of zero have been overcome. With slight modifications, the present proofs are applicable to algebras.

A *non-associative ring* $\mathfrak{R}$ is an additive abelian group closed under a product operation with respect to which the two distributive laws hold. Multiplication on the right (left) by a fixed element $x \in \mathfrak{R}$ determines an endomorphism R_x (L_x) of $\mathfrak{R}$ as an additive group. For $x, y \in \mathfrak{R}$,

$$x \cdot y = xR_y = yL_x.$$

The R_x and L_y generate an associative ring $\mathfrak{A}$ called the *transformation ring* of $\mathfrak{R}$. Clearly $\mathfrak{R}$ can be construed as a representation space for $\mathfrak{A}$, and this representation is faithful. The two-sided ideals of $\mathfrak{R}$, which are defined as for associative rings, are exactly the $\mathfrak{A}$-subspaces of $\mathfrak{R}$. The theory of ring homomorphisms goes over intact to the non-associative case.

A nonzero element $a \in \mathfrak{R}$ is called an *absolute divisor of zero* if $a \cdot x = x \cdot a = 0$ for all $x \in \mathfrak{R}$. If $\mathfrak{A}$ has a unit element I and $\mathfrak{R}$ contains no absolute divisors of zero, then the unit element I is the identity

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[1] Numbers in brackets refer to the references cited at the end of the paper.

mapping on $\mathfrak{R}$. For if $aI=b$ where $a, b\in\mathfrak{R}$, then $aI-b=0=(aI-b)I=(a-b)I$, $(a-b)\mathfrak{A}=0$ and so $a=b$.

We now assume that the minimum condition holds in $\mathfrak{A}$. It has been shown in [5, Theorems 3 and 14] that for the case of simple rings this is equivalent to the assumption that $\mathfrak{R}$ can be regarded as an algebra of finite dimension over a certain field.

THEOREM 1. *Let $\mathfrak{A}$ satisfy the minimum condition and $\mathfrak{R}$ contain no absolute divisors of zero. Then $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$, where the $\mathfrak{R}_i$ are simple if and only if $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ where the $\mathfrak{A}_i$ are simple, and conversely.*

PROOF. If $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$ where the $\mathfrak{R}_i$ are simple, then $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ where $\mathfrak{A}_i$ is the transformation ring for $\mathfrak{R}_i$. Since $\mathfrak{R}_i$ is an irreducible $\mathfrak{A}_i$-space, it follows that $\mathfrak{A}_i$ is simple. Conversely, suppose $\mathfrak{A}=\mathfrak{A}_1\oplus\mathfrak{A}_2\oplus\cdots\oplus\mathfrak{A}_n$ where the $\mathfrak{A}_i$ are simple rings generated by pairwise orthogonal idempotents E_i in the center of $\mathfrak{A}$. Clearly $\mathfrak{R}=\mathfrak{R}_1\oplus\mathfrak{R}_2\oplus\cdots\oplus\mathfrak{R}_n$, where $\mathfrak{R}_i=\mathfrak{R}E_i$. It is easily seen that $\mathfrak{A}_i$ is the transformation ring for $\mathfrak{R}_i$. If $\mathfrak{R}_i$ has a proper ideal $\mathfrak{M}$, then $\mathfrak{R}_i=\mathfrak{M}\oplus\mathfrak{N}$, a direct sum of $\mathfrak{A}_i$-subspaces since $\mathfrak{A}_i$ is semi-simple. Let $\mathfrak{S}$ be the ideal of $\mathfrak{A}_i$ which annihilates $\mathfrak{M}$. For every nonzero element x in $\mathfrak{N}$, R_x and L_x, at least one of which must be nonzero, are in $\mathfrak{S}$. Clearly E_i is not in $\mathfrak{S}$ and so $\mathfrak{S}$ is a proper ideal of $\mathfrak{A}_i$, which is impossible. Therefore $\mathfrak{R}_i$ is simple.

COROLLARY. *Under the assumptions of Theorem* 1, *$\mathfrak{R}$ is simple if and only if $\mathfrak{A}$ is simple.*

The ring $\mathfrak{R}$ is defined to be *semisimple* if the following conditions are satisfied:

(i) $\mathfrak{A}$ satisfies the minimum condition.

(ii) $\mathfrak{R}$ contains no absolute divisors of zero.

(iii) $\mathfrak{R}$ is a direct sum of simple rings.

We now eliminate the restriction that $\mathfrak{R}$ contain no absolute divisors of zero. Suppose $\mathfrak{R}$ has an ideal $\mathfrak{H}$. Let $\mathfrak{S}$ be the set consisting of all $S\in\mathfrak{A}$ such that $\mathfrak{R}S\subseteq\mathfrak{H}$. Clearly $\mathfrak{R}\mathfrak{A}\mathfrak{S}\subseteq\mathfrak{R}\mathfrak{S}\subseteq\mathfrak{H}$ and $\mathfrak{R}\mathfrak{S}\mathfrak{A}\subseteq\mathfrak{H}\mathfrak{A}\subseteq\mathfrak{H}$. Furthermore, $\mathfrak{S}$ is an additive subgroup and so an ideal of $\mathfrak{A}$. Let $[x]$ and $[y]$ be any two residue classes of $\mathfrak{R}-\mathfrak{H}$. Then $[x]\cdot[y]=[x\cdot y]=[xR_y]=[yL_x]$. If $[xU]=[xV]$ for all $[x]\in\mathfrak{R}-\mathfrak{H}$ where $U, V\in\mathfrak{A}$, then $U-V\in\mathfrak{S}$ and so right (left) multiplication by a fixed element of $\mathfrak{R}-\mathfrak{H}$ determines a unique residue class of $\mathfrak{A}-\mathfrak{S}$. Then it is easy to prove that $\mathfrak{A}-\mathfrak{S}$ is isomorphic to the transformation ring for $\mathfrak{R}-\mathfrak{H}$. It follows immediately that if $\mathfrak{A}$ satisfies the minimum

condition, then so does the transformation ring for any difference ring of $\mathfrak{R}$.

Let $\mathfrak{M}_1$ be the ideal consisting of zero and all absolute divisors of zero in $\mathfrak{R}$. Continuing by induction, let $\mathfrak{M}_{i+1}$ be the set consisting of all $x \in \mathfrak{R}$ such that $a \cdot x$ and $x \cdot a \in \mathfrak{M}_i$ for all $a \in \mathfrak{R}$. The $\mathfrak{M}_i$ are ideals of $\mathfrak{R}$ and $\mathfrak{M}_i \subseteq \mathfrak{M}_{i+1}$.[2]

LEMMA 1. *$\mathfrak{M}_{j+1}$ is the set consisting of all $x \in \mathfrak{R}$ such that R_x and L_x are left-annihilators of $\mathfrak{A}^j$.*

PROOF. Suppose $x \in \mathfrak{M}_{j+1}$. Then aR_x and $aL_x \in \mathfrak{M}_j$ for all $a \in \mathfrak{R}$. Each succeeding application of transformation in $\mathfrak{A}$ gives rise to an element in the preceding $\mathfrak{M}_i$. Clearly $aR_xT_1T_2 \cdots T_j = aL_xT_1T_2 \cdots T_j = 0$ for all $a \in \mathfrak{R}$, where $T_1, T_2, \cdots, T_j$ are arbitrary elements of $\mathfrak{A}$. Therefore $R_xT_1T_2 \cdots T_j = L_xT_1T_2 \cdots T_j = 0$ and so R_x and L_x are left annihilators of $\mathfrak{A}^j$. Conversely, if $aR_xT_1T_2 \cdots T_j = aL_xT_1T_2 \cdots T_j = 0$ for all $a \in \mathfrak{R}$, where $T_1, T_2, \cdots, T_j$ are arbitrary elements of $\mathfrak{A}$, then $aR_xT_1T_2 \cdots T_{j-1}$ and $aL_xT_1T_2 \cdots T_{j-1}$ are in $\mathfrak{M}_1$ by definition of $\mathfrak{M}_1$. Continuing in this way, it is easily seen that aR_x and aL_x, that is, $a \cdot x$ and $x \cdot a$ are in $\mathfrak{M}_j$ for all $a \in \mathfrak{R}$ and so $x \in \mathfrak{M}_{j+1}$.

LEMMA 2. *If $\mathfrak{A}$ satisfies the minimum condition, then there exists a least integer l such that $\mathfrak{M}_l = \mathfrak{M}_{l+1}$.*

PROOF. Let k be the first integer for which $\mathfrak{A}^{k-1} = \mathfrak{A}^k$. The existence of k is insured by the minimum condition on $\mathfrak{A}$. Suppose x is an element of $\mathfrak{M}_{k+1}$ not in $\mathfrak{M}_k$. By Lemma 1, R_x and L_x would be left-annihilators of $\mathfrak{A}^k$ but not both of $\mathfrak{A}^{k-1}$. This is impossible, and so $\mathfrak{M}_{k+1} = \mathfrak{M}_k$. Therefore k is an upper bound for l and the lemma is proved.

In particular, if $\mathfrak{A}$ has a unit element, then $\mathfrak{M}_2 = \mathfrak{M}_1$, since $\mathfrak{A}$ can have no left-annihilators.

THEOREM 2. *Let $\mathfrak{A}$ satisfy the minimum condition. Then $\mathfrak{R} - \mathfrak{M}_l$ has no absolute divisors of zero. Furthermore, $\mathfrak{M}_l$ is contained in every ideal $\mathfrak{H}$ for which $\mathfrak{R} - \mathfrak{H}$ has no absolute divisors of zero.*

PROOF. If $a \cdot x$ and $x \cdot a \in \mathfrak{M}_l$ for all $a \in \mathfrak{R}$, then $x \in \mathfrak{M}_{l+1} = \mathfrak{M}_l$. This proves the first part. Suppose $\mathfrak{R} - \mathfrak{H}$ has no absolute divisors of zero. Clearly $\mathfrak{M}_1 \subseteq \mathfrak{H}$. Now assume $\mathfrak{M}_i \subseteq \mathfrak{H}$. If $x \in \mathfrak{M}_{i+1}$, then both $a \cdot x$ and $x \cdot a \in \mathfrak{M}_i \subseteq \mathfrak{H}$ for all $a \in \mathfrak{R}$ and so $x \in \mathfrak{H}$. Therefore $\mathfrak{M}_l \subseteq \mathfrak{H}$.

LEMMA 3. *If $\mathfrak{A}$ satisfies the minimum condition and $\mathfrak{R} - \mathfrak{H}$ is semi-*

[2] In the case of Lie rings, the $\mathfrak{M}_i$ constitute the upper central chain.

simple, then $\mathfrak{RR} \subseteq \mathfrak{H}$ *where* $\mathfrak{R}$ *is the radical of* $\mathfrak{A}$.

PROOF. The transformation ring for $\mathfrak{R} - \mathfrak{H}$ is isomorphic to $\mathfrak{A} - \mathfrak{S}$ where $\mathfrak{S}$ is the ideal consisting of all $S \in \mathfrak{A}$ such that $\mathfrak{R}S \subseteq \mathfrak{H}$. By Theorem 1, $\mathfrak{A} - \mathfrak{S}$ is semisimple. Therefore $\mathfrak{R} \subseteq \mathfrak{S}$, and so $\mathfrak{RR} \subseteq \mathfrak{RS} \subseteq \mathfrak{H}$.

We now consider the ideal $\mathfrak{M}_l$ for the ring $\mathfrak{R} - \mathfrak{RR}$. The existence of this ideal follows from Lemma 2 since the transformation ring for $\mathfrak{R} - \mathfrak{RR}$ satisfies the minimum condition. Let $\mathfrak{M}$ be the complete reciprocal image of $\mathfrak{M}_l$ under the natural homomorphism $\mathfrak{R} \to \mathfrak{R} - \mathfrak{RR}$. The ideal $\mathfrak{M}$ will be call d the *radical* of $\mathfrak{R}$. Since $\mathfrak{R} - \mathfrak{M} \cong (\mathfrak{R} - \mathfrak{RR}) - \mathfrak{M}_l$, it follows from Theorem 2 that $\mathfrak{R} - \mathfrak{M}$ has no absolute divisors of zero. The transformation ring for $\mathfrak{R} - \mathfrak{M}$ is isomorphic to $\mathfrak{A} - \mathfrak{S}$, where $\mathfrak{S}$ is the ideal consisting of all $S \in \mathfrak{A}$ such that $\mathfrak{R}S \subseteq \mathfrak{M}$. Clearly $\mathfrak{R} \subseteq \mathfrak{S}$ and so $\mathfrak{A} - \mathfrak{S}$ is semisimple. Theorem 1 then implies

THEOREM 3. *Let* $\mathfrak{A}$ *satisfy the minimum condition. Then* $\mathfrak{R} - \mathfrak{M}$ *is semisimple.*

That $\mathfrak{M}$ is the minimal ideal having this property is shown by

THEOREM 4. *If* $\mathfrak{A}$ *satisfies the minimum condition and* $\mathfrak{R} - \mathfrak{H}$ *is semisimple for some ideal* $\mathfrak{H}$, *then* $\mathfrak{M} \subseteq \mathfrak{H}$.

PROOF. By Lemma 3, $\mathfrak{RR} \subseteq \mathfrak{H}$. Clearly $(\mathfrak{R} - \mathfrak{RR}) - (\mathfrak{H} - \mathfrak{RR}) \cong \mathfrak{R} - \mathfrak{H}$ and so by Theorem 2, $\mathfrak{M}_l \subseteq \mathfrak{H} - \mathfrak{RR}$, which implies $\mathfrak{M} \subseteq \mathfrak{H}$.

The writer is indebted to D. C. Murdoch for suggesting Lemma 1.

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SOME REMARKS ON ONE-SIDED INVERSES

N. JACOBSON

Let $\mathfrak{A}$ be an arbitrary ring with an identity 1, and suppose that $\mathfrak{A}$ contains a pair of elements u, v such that

$$uv = 1 \text{ but } vu \neq 1. \tag{1}$$

We introduce the elements

$$e_{i,j} = v^{i-1}u^{j-1} - v^i u^j \tag{2}$$

for $i, j = 1, 2, 3, \cdots$, where it is understood that $u^0 = 1 = v^0$. It can be verified directly that the e_{ij} thus defined satisfy the multiplication table for matrix units:

$$e_{ij}e_{rs} = \delta_{jr}e_{is}. \tag{3}$$

In particular the elements $e_i = e_{ii}$ are orthogonal idempotent elements. No $e_{ij} = 0$. For by (3) the vanishing of one of the e_{ij} implies the vanishing of all; in particular, it implies that

$$0 = e_1 = 1 - vu$$

contrary to (1).

The existence of an infinite set of orthogonal idempotent elements in a ring $\mathfrak{A}$ is incompatible with mild chain conditions on the ring. If $\{e_i\}$ is such a set of idempotent elements and we set $f_k = \sum_1^k e_i$, then

$$f_1\mathfrak{A} \subset f_2\mathfrak{A} \subset f_3\mathfrak{A} \subset \cdots \tag{4}$$

is an infinite properly ascending chain of right ideals. The right annihilator of an idempotent element f is the set of elements $\{a - fa\}$. If $\mathfrak{A}$ has an identity, this right ideal is the principal right ideal $(1-f)\mathfrak{A}$. Even if $\mathfrak{A}$ does not have an identity, it is customary to denote the set $\{a - fa\}$ as $(1-f)\mathfrak{A}$. It is clear that the following is an infinite properly descending chain of annihilators

$$(1 - f_1)\mathfrak{A} \supset (1 - f_2)\mathfrak{A} \supset (1 - f_3)\mathfrak{A} \supset \cdots. \tag{5}$$

Our remarks imply the following theorem which includes a result due to Baer.[1]

THEOREM 1. *If $\mathfrak{A}$ is a ring with an identity that satisfies either the*

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[1] R. Baer, *Inverses and zero-divisors*, Bull. Amer. Math. Soc. vol. 48 (1942) pp. 630–638.

ascending or the descending chain condition for principal right ideals generated by idempotent elements, then $uv=1$ *in* $\mathfrak{A}$ *implies* $vu=1$.

If $e_{11}, e_{12}, e_{21}, e_{22}$ are elements of a ring satisfying (3), then $e_{11}\mathfrak{A}=e_{12}\mathfrak{A}$ and the mapping $x\to e_{12}x$ is an $\mathfrak{A}$-isomorphism of $e_{22}\mathfrak{A}$ onto $e_{11}\mathfrak{A}$. Hence if the e_{ii} are defined as above, then the right ideals $e_{ii}\mathfrak{A}$ are $\mathfrak{A}$-isomorphic. The right ideal $\mathfrak{B}=\sum e_{ii}\mathfrak{A}$ is a direct sum of the $e_{ii}\mathfrak{A}$. Hence we have the following result.

THEOREM 2. *If* $\mathfrak{A}$ *is a ring with an identity that contains two elements* u *and* v *such that* $uv=1$, $vu\neq 1$, *then* $\mathfrak{A}$ *contains a right ideal that is a direct sum of an infinite number of* $\mathfrak{A}$*-isomorphic right ideals.*

We note next a result that was proved first by Kaplansky (oral communication) using structure theory.

THEOREM 3. *If an element of a ring with an identity has more than one right inverse, then it has an infinite number of right inverses.*

PROOF. If v is one of the right inverses of the element u, then we have $uv=1$, $vu\neq 1$. If the e_{ij} are defined as above, then $ue_{11}=u(1-vu)=0$. Hence also $ue_{1k}=0$ for $k=1, 2, 3, \cdots$. If $e_{1k}=e_{1l}$ for $k\neq l$, then $e_{1k}e_{kk}=e_{1l}e_{kk}$ and $e_{1k}=0$. Hence the e_{1k} are all different and the elements $v_k=v+e_{1k}$ are all different. Evidently $uv_k=1$.

We assume next that $\mathfrak{A}$ is an algebra over a field Φ with an identity and that $\mathfrak{A}$ contains elements u, v satisfying (1). We wish to determine the structure of the algebra $\Phi[u, v]$ generated by u and v. For this purpose we introduce a vector space $\mathfrak{R}$ that has a denumerable basis $(x_1, x_2, x_3, \cdots)$ over Φ. Let $\overline{U}$ and $\overline{V}$, respectively, be the linear transformations in $\mathfrak{R}$ that have the matrices

$$(6)\qquad U=\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ 0 & 0 & 0 & 1 & \\ & & & \ddots & \ddots \end{pmatrix}, \qquad V=\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix}$$

relative to the given basis. We have the relations $UV=1$, $VU\neq 1$. Hence any element in $\Phi[U, V]$ is a linear combination of the elements V^iU^j, $i, j=0, 1, 2, \cdots$. We shall now show that these elements are linearly independent; hence they form a basis. We can verify that

$$(7)\qquad V^iU^i=\operatorname{diag}\{0, 0, \cdots, 0; 1, 1, \cdots\},$$

where there are i zeros. Hence V^iU^{i+k} has nonzero elements only in the kth super-diagonal and $V^{i+k}U^i$ has nonzero elements only in the kth sub-diagonal. Any relation $\sum\beta_{ij}\ V^iU^j=0$ therefore implies that

$$\sum_i \beta_{i,i+k}V^iU^{i+k} = 0, \qquad \sum_i \beta_{i+k,i}V^{i+k}U^i = 0.$$

Multiplication of the first of these equations on the right by V^k gives

$$\sum \beta_{i,i+k}V^iU^i = 0.$$

It is evident from (7) that the matrices $1=V^0U^0,\ V^1U^1, \cdots$ are linearly independent. Hence $\beta_{i,i+k}=0$. Similarly every $\beta_{i+k,i}=0$. This proves our assertion.

The matrices

$$E_{i,j} = (V^{i-1}U^{j-1} - V^iU^j) \tag{8}$$

are the usual matrix units. Hence $\Phi[U,\ V]$ contains every matrix of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \tag{9}$$

where A is a finite square matrix. Using (8), we can express any V^iU^j in the form $\sum\beta_{rs}E_{rs}+\phi(U)+\psi(V)$, ϕ and ψ polynomials.

Now it is clear that the subalgebra of $\Phi[\overline{U},\ \overline{V}]$ corresponding to the algebra of matrices (9) is a dense algebra of linear transformations of finite rank.[2] Also it is easy to see that $\phi(\overline{U})+\psi(\overline{V})$ has infinite rank unless ϕ and ψ are 0. Hence the transformations with matrices (9) constitute the complete set of linear transformations of finite rank in $\Phi[U,\ V]$. It follows from known structure results that $\Phi[\overline{U},\ \overline{V}]$ (and $\Phi[U,\ V]$) is a primitive algebra that has minimal one-sided ideals.[3] Moreover, the subalgebra corresponding to (9) is the minimal two-sided ideal of this algebra. Any nonzero two-sided ideal contains this one, and in particular it contains the elements $1-\overline{V}\,\overline{U}$.

Since the linear transformations $\overline{V}^i\overline{U}^j$ are linearly independent it is clear that the mapping $\overline{U}\to u,\ \overline{V}\to v$ can be extended to a homomorphism of $\Phi[\overline{U},\ \overline{V}]$ onto $\Phi[u,\ v]$. Since $vu\neq 1$, the kernel of this homomorphism does not include $1-\overline{V}\,\overline{U}$. Hence it is 0 and our correspondence is an isomorphism. This completes the proof of the following theorem:

THEOREM 4. *Any two algebras* $\Phi[u_i,\ v_i]$, $i=1,\ 2$, *in which*

[2] Cf. the author's, *The radical and semi-simplicity for arbitrary rings*, Amer. J. Math. vol. 47 (1945) p. 313.

[3] See the reference cited in footnote 2, p. 317.

$u_iv_i=1$, $v_iu_i\neq 1$ are isomorphic under an isomorphism that pairs the u_i and the v_i. The algebras $\Phi[u_i, v_i]$ are primitive algebras that have minimal one-sided ideals.

The minimal two-sided ideal $\mathfrak{B}$ of $\Phi[u, v]$ is the infinite matrix algebra with basis e_{ij}. Any element of $\Phi[u, v]$ is congruent mod $\mathfrak{B}$ to an element of the form $\phi(u)+\psi(v)$. It follows that $\Phi[u, v]/\mathfrak{B}$ is isomorphic to the group algebra of an infinite cyclic group.

Suppose now that $\mathfrak{R}$ is any vector space over Φ and that $\overline{U}$ and $\overline{V}$ are linear transformations in $\mathfrak{R}$ over Φ such that $\overline{U}\overline{V}=1$, $\overline{V}\overline{U}\neq 1$. Let $\overline{\mathfrak{B}}$ denote the minimal two-sided ideal in $\Phi[\overline{U}, \overline{V}]$. Then $\overline{\mathfrak{B}}$ has the basis $\overline{E}_{ij}$ defined as in (2). It follows easily that $\overline{\mathfrak{B}}$ is a direct sum of the right ideals $\overline{\mathfrak{J}}_k=\overline{E}_{kk}\Phi[\overline{U}, \overline{V}]$ and that the $\overline{\mathfrak{J}}_k$ are minimal: If $x\in\mathfrak{R}$, the subspace $x\overline{\mathfrak{J}}_k$ is either 0 or it is $\Phi[\overline{U}, \overline{V}]$-isomorphic to $\overline{\mathfrak{J}}_k$. In the latter case $x\overline{\mathfrak{J}}_k$ is irreducible. It follows that the subspace $\mathfrak{R}\overline{\mathfrak{B}}$ can be decomposed as a direct sum of subspaces that are invariant and irreducible relative to $\overline{U}$ and $\overline{V}$. It is easy to see that all of these spaces are isomorphic and that if suitable bases are chosen in these spaces, then the matrices U and V have the form (6). The factor space $\mathfrak{S}=\mathfrak{R}-\mathfrak{R}\overline{\mathfrak{B}}$ is annihilated by $\overline{\mathfrak{B}}$. Hence the induced transformations $\overline{U}$ and $\overline{V}$ in this space satisfy $\overline{U}\overline{V}=1=\overline{V}\overline{U}$.

Nearly all of our results hold also for quasi-inverses. In any ring $\mathfrak{A}$ we define $a\circ b=a+b-ab$. Then $\mathfrak{A}$ is a semigroup relative to this composition and 0 is the identity. An element b is a *right quasi-inverse* of a if $a\circ b=0$. If $\mathfrak{A}$ has an identity 1, $(1-a)(1-b)=1-a\circ b$, so that if $a\circ b=0$, then $(1-a)(1-b)=1$ and conversely. Now suppose that $\mathfrak{A}$ contains two elements a and b such that

$$a\circ b=0, \qquad b\circ a\neq 0. \tag{10}$$

If we define $x^{\circ k}=x^{\circ k-1}\circ x$, then we can verify that the elements

$$e_{i,j}=b^{\circ i}\circ a^{\circ j}-b^{\circ i-1}\circ a^{\circ j-1} \tag{11}$$

satisfy the multiplication for matrix units. All of the e_{ij} are nonzero. In particular, $\mathfrak{A}$ contains an infinite number of orthogonal idempotent elements. Then we see that if $\mathfrak{A}$ satisfies the ascending chain condition on principal ideals generated by idempotent elements, then $a\circ b=0$ in $\mathfrak{A}$ implies $b\circ a=0$.[4] This is the analogue of Theorem 1. Theorems 2, 3, and 4 carry over without change.

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[4] Baer's results cited in footnote 1 have been extended to quasi-inverses by Andrunakievic in his paper *Semi-radical rings*, Izvestiya Akademii Nauk SSSR. Ser. Mat. vol. 12 (1948) pp. 129–178.

THE WEIERSTRASS THEOREM IN FIELDS WITH VALUATIONS

IRVING KAPLANSKY[1]

In [2, Theorem 32][2] the author showed that an analogue of the Weierstrass-Stone theorem holds in topological rings having ideal neighborhoods of 0. Earlier, Dieudonné [1] had proved the Weierstrass-Stone theorem for the field of p-adic numbers. Now the field of p-adic numbers has an open subring (the p-adic integers) with ideal neighborhoods of 0. It seems plausible, therefore, to expect that the method of [2] will apply, provided one has a supplementary device for "getting into" the p-adic integers. This is in fact the case, and the result is applicable to any division ring with a valuation of rank one.[3] The requisite lemma reads as follows:

LEMMA 1. *Let F be a division ring with a valuation of rank one, and B its valuation ring. Let a be any nonzero element in F, and K a compact subset of F. Then there exists a (non-commutative) polynomial f with coefficients in F, without a constant term, and satisfying $f(a)=1$, $f(K)\subset B$.*

PROOF. Let P denote the maximal ideal in B, that is, the set of all elements x with $V(x)>0$, where V denotes the valuation. Let K' denote the subset of K with values less than $V(a)$; K' will again be compact. For any c in K' there is a compact open subset of K' containing c and contained in $c(1+P)$. Take a finite covering of K' consisting of such neighborhoods: say $U_1, \cdots, U_r$ with U_i contained in $c_i(1+P)$. Suppose the c's numbered so that $V(c_i)\geqq V(c_{i+1})$. Now $1-c_i^{-1}U_i$ is a compact subset of P, and consequently the values of its elements have a positive lower bound α_i. Choose integers $n(1), \cdots, n(r)$ in succession large enough so that

$$V(a^{-1}c_i) + \sum_{j=1}^{i-1} n(j)V(c_j^{-1}c_i) + n(i)\alpha_i \geqq 0$$

for $i=1, \cdots, r$. Then the polynomial

$$f(x) = 1 - (1 - a^{-1}x)(1 - c_1^{-1}x)^{n(1)} \cdots (1 - c_r^{-1}x)^{n(r)}$$

satisfies the requirements of the lemma.

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[1] John Simon Guggenheim Memorial Fellow.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] That is to say, a valuation whose value group is a subgroup of the real numbers.

We shall restate [2, Theorem 32] in a slightly sharpened form.

LEMMA 2. *Let A be a topological ring with unit element and ideal neighborhoods of 0, let X be a totally disconnected locally compact Hausdorff space, and C the ring of all continuous functions from X to A vanishing at ∞. Topologize C by uniform convergence. Let D be a closed subring of C, containing for any distinct points $x, y \in X$ and any $a, b \in A$, a function f with $f(x)=a$, $f(y)=b$. Then $D=C$.*

PROOF. Let U be a fixed ideal neighborhood of 0, $1 \notin U+U$. Let K be a compact subset of X and x a point not in K. Then D contains a function f with $f(x)=1$ and $f(y)=0$ for a given y in K. The function f will take values in U in a suitable neighborhood of y. A finite number of these neighborhoods cover K; if g is the product of the corresponding f's, we have $g(x)=1$, $g(K) \subset U$.

We start again with an arbitrary $z \in X$ and an h in D with $h(z)=1$. There is a compact neighborhood L of z with $h(L) \subset 1+U$, and a larger compact set M such that h is in U in the complement of M. For any given w in $M-L$ we can, by the preceding paragraph, find a function p in D with $p(L) \subset U$, $p(w)=h(w)$. Then the function $h(h-p)$ has the following properties: in L its values lie in $1+U$, and it vanishes at w and accordingly lies in U in a neighborhood of w. A finite number of these neighborhoods cover the compact set $M-L$. The product of the corresponding elements $h(h-p)$ gives us an element q in D with $q(L) \subset 1+U$, $q(L') \subset U$, L' the complement of L. By combining such elements we can get "within U" of the characteristic function of any compact open set. Since D is closed, it actually contains all characteristic functions of compact open sets, from which it follows readily that $D=C$.

As an immediate consequence of Lemmas 1 and 2, we have the following generalization of Dieudonné's theorem:

THEOREM. *Let F be a division ring with a valuation of rank one, X a totally disconnected locally compact Hausdorff space, C the ring of all continuous functions from X to F vanishing at ∞. Topologize C by uniform convergence. Let D be a closed subring of C, admitting left-multiplication by the constant functions, and containing for any two distinct points $x, y \in X$ a function vanishing at x but not at y. Then $D=C$.*

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SOME CHARACTERS OF THE SYMMETRIC GROUP

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Introduction. Frobenius [1][1] derived expressions for the characters of a few very simple classes of S_m, the symmetric group on m things. Here we give formulas for some more complicated classes. The method is quite general. A recurrence formula, due to Murnaghan, is used. The need for the formulas arose from general considerations of nuclear binding forces.[2] Some applications will be given in another paper.

1. The character $\chi(p)_\lambda$, of the class $\alpha_1=m-p$, $\alpha_p=1$ [1, 2]. If $\{\lambda\}$, in the usual notation, is a partition of m, the same partition may also be represented [6] by two sets of numbers $\{b, a\}$ and the following relation holds,

$$(1.1)\quad \sum_{j=1}^{b}(\lambda_j-j)^n=\sum_{j=1}^{b}b_j^n+(-1)^{n+1}\sum_{j=1}^{a}(a_j+1)^n+(-1)^n\sum_{j=1}^{k}j^n.$$

If χ_λ is the character of a class K of S_m,

$$\{\lambda_j'\}=\{\lambda_1,\lambda_2,\cdots,\lambda_j-p,\cdots,\lambda_k\}$$

a partition of $m-p$, and $\chi_{\lambda_j'}$ the character of the class K' of S_{m-p}, where K' has the same cyclic structure as K, but with one less p-cycle, then

$$(1.2)\qquad \chi_\lambda=\sum_{j=1}^{k}\chi_{\lambda_j'}\qquad \text{(Murnaghan Recurrence Formula [6]).}$$

Denoting by $\chi(p)_\lambda$ the character of the class having one p-cycle (all others unary), we have

$$\chi(p)_\lambda=\sum_{1}^{k}\chi_{\lambda_j'}=\sum_{1}^{k}D_{\lambda_j'}$$

where D is the dimension of the representation.

$$D_{\lambda_j'}=\frac{(m-p)!\Delta(l_j')}{l_1!l_2!\cdots(l_j-p)!\cdots l_k!},$$

$$\left(\Delta(l)=\prod_{r<s}(l_r-l_s),\ l_r=\lambda_r+k-r\right)$$

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[1] Numbers in brackets refer to bibliography at end of this paper.

[2] The original problem to find $\chi(2, 2)_\lambda$ is due to Dr. J. A. Wheeler, Princeton [7].

$$= \frac{(m-p)!}{m!} \frac{l_j!}{(l_j-p)!} \frac{f(l_j-p)}{-pf'(l_j)} \qquad \left(\text{where } f(x) = \prod_{i=1}^{k}(x-b_i)\right)$$

$$= \frac{(m-p)!}{m!} D_\lambda \left[x(x-1)\cdots(x-p+1) \frac{f(x-p)}{f(x)} \right]_{x=l_j}.$$

This expression for $D_{\lambda'_j}$ may be written as $\sum_1^k A_j/(x-\lambda_j)$ plus a polynomial in x, where $A_j = l_j(l_j-1)\cdots(l_j-p+1)f(l_j-p)/f'(l_j)$. If we expand $\sum_1^k A_j/(x-l_j)$ in descending powers of x, it is evident that $\sum_1^k A_j$ is the coefficient of $1/x$ in the expansion of

$$x(x-1)\cdots(x-p+1)\frac{f(x-p)}{f(x)} \tag{1.3}$$

in descending powers of x.

Likewise

$$\sum_1^k l_j A_j \qquad \text{is the coefficient of } 1/x^2 \tag{1.4}$$

and

$$\sum_1^k l_j^\alpha A_j \qquad \text{is the coefficient of } 1/x^{\alpha+1}. \tag{1.5}$$

Thus

$$\sum_1^k l_j^\alpha D_{\lambda'_j} = -\frac{m-p!}{m!} D_\lambda \frac{1}{p} \sum_1^k l_j^\alpha A_j$$

and taking $\alpha = 0$, we have[3]

$$\chi(p)_\lambda = \sum_1^k D_{\lambda'_j} = \frac{m-p!}{m!} D_\lambda \left(\frac{-1}{p}\right) \sum_1^k A_j,$$

$$\frac{m!}{m-p!} \frac{\chi(p)_\lambda}{D_\lambda} = \left[x(x-1)\cdots(x-p+1) \frac{f(x-p)}{-pf(x)} \right]_{1/x}. \tag{1.6}$$

It should be noted that if $\{\lambda'_j\}$ ends in a negative number when arranged in non-ascending order [6], then $\lambda_j - p + k - j < 0$, as $\lambda_j - p$ may be increased by $k-j$ by moving down to kth position from jth position. Therefore

$$l_j - p < 0,$$

[3] $[\ ]_{1/x}$ signifies the coefficient of $1/x$ in the expansion of the function within the brackets.

$$l_j - p + 1 \leqq 0.$$

Therefore

$$A_j = 0.$$

If $\{\lambda'_j\}$ cannot be rearranged to form a non-increasing sequence

$$\lambda_j - p = \lambda_{j+s} - S, \qquad \text{some } S,$$
$$\lambda_j + j - K - p = \lambda_{j+s} + j - K - S,$$
$$l_j - p = l_{j+s}.$$

Therefore

$$f(l_j - p) = 0, A_j = 0.$$

In neither case is any contribution made to $\sum_1^k A_j$.

Let $x = y + k'$, then

$$\left[x(x-1)\cdots(x-p+1)\frac{f(x-p)}{f(x)}\right]_{1/x} = \left[y(y-1)\cdots(y-p+1)\frac{F(y-p)}{F(y)}\right]_{1/y} \tag{1.7}$$

where

$$F(y) = \prod_{i=1}^{s}\left[\frac{(y-b_i)}{(y+a_i+1)}\right], \qquad a, b \text{ as in (1.1)},$$

and

$$\begin{aligned}\frac{F(y-p)}{-pF(y)} = &-1/p + \frac{m_2}{y} + \frac{1}{y^2}(c_2 + pm) \\ &+ \frac{1}{y^4}\left(c_4 + \frac{3}{2}pc_2 + p^2m - \frac{p}{2}m^2\right) \\ &+ \frac{1}{y^5}(c_5 + 2pc_4 - pmc_2 + 2p^2c_2 - p^2m^2 + p^3m) \\ &+ \frac{1}{y^6}\left(c_6 - \frac{p}{2}c_2^2 - pmc_4 + \frac{5}{2}pc_3 - \frac{5}{2}p^2mc_2\right. \\ &\left. + \frac{10}{3}p^2c_4 + \frac{p^2m^2}{6} - \frac{3}{2}m^2p^2 + \frac{5}{2}p^3c_2 + mp^4\right) \\ &+ \cdots\end{aligned} \tag{1.8}$$

with

$$m = \sum_1^s \{b_j + (a_j + 1)\},$$

$$c_{2r+1} = \sum_1^s \{b_j^{2r} - (a_j + 1)^{2r}\},$$

$$c_{2r} = \sum_1^s \{b_j^{2r+1} + (a_j + 1)^{2r}\}.$$

Using (1.7) and (1.8) in (1.6), we get the expressions (5.1), (5.2), (5.3), (5.4), (5.5) on pp. 367–368.

2. **The character $\chi(p, 2)_\lambda$ of the class $\alpha_1 = m-p-2$, $\alpha_2 = 1$, $\alpha_p = 1$.** Let $\chi(p, 2)_\lambda$ be the character of the class $\alpha_2 = 1$ binary cycle and $\alpha_p = 1$ cycle on p letters.

By the recurrence formula, (1.2),

$$\chi(p, 2)_\lambda = \sum_{j=1}^k \chi(2)_{\lambda_j'} \tag{2.1}$$

where λ_j', as a subscript, indicates that $\chi(2)$ is the character of the representation $\{\lambda_j'\} = \{\lambda_1, \lambda_2, \cdots, \lambda_j - p, \cdots, \lambda_k\}$ of the symmetric group on $m-p$ letters, S_{m-p}, corresponding to the class $\alpha_1 = m-p-2$ unary cycles, $\alpha_2 = 1$ binary cycle. By (5.2),

$$\chi(2)_{\lambda_j'} = (M_2)_j' D_{\lambda_j}(m-p-2)!/(m-p)!$$

where $(M_2)_j'$ is M_2 with $\lambda_j - p$ instead of λ_j (that is, the correct value of M_2 for $\{\lambda_j'\}$).

$D_{\lambda_j'}$ is the dimension of $\{\lambda_j'\}$, and $m-p$ replaces m as $\{\lambda_j'\}$ is a representation of S_{m-p}.

$$M_2 = \sum_{j=1}^k (\lambda_j - j)(\lambda_j - j + 1) - \sum_1^k j^2 + \sum_1^k j$$

is a function of $\lambda_1, \lambda_2, \cdots, \lambda_k$.

$(M_2)_j'$ is the value for $\lambda_1, \lambda_2, \cdots, (\lambda_j - p), \cdots, \lambda_k$,

$$(M_2)_j' = M_2 - 2p(\lambda_j - j) + p(p-1).$$

Therefore

$$\chi(2)_{\lambda_j'} = \frac{M_2 + p(p-1) - 2p(\lambda_j - j)}{(m-p)(m-p-1)} D_{\lambda_j'}.$$

Then, (2.1),

$$\chi(p,2)_\lambda = \sum_1^k \frac{M_2 + p(p-1) - 2p(\lambda_j - j)}{(m-p)(m-p-1)} D_{\lambda_j'}$$

$$= \frac{M_2 + p(p-1)}{(m-p)(m-p-1)} \sum_1^k D_{\lambda_j'} \tag{2.2}$$

$$- \frac{2p}{(m-p)(m-p-1)} \sum_1^k (\lambda_j - j) D_{\lambda_j'}$$

(since M_2, p, m are the same in all terms). $\chi(p)_\lambda = \sum_1^k D_{\lambda_j}$ has already been found.

$$\sum_1^k (\lambda_j - j) D_{\lambda_j'} = \sum_1^k (l_j - k) D_{\lambda_j'}$$

$$= \sum_1^k l_j D_{\lambda_j'} - k \sum_1^k D_{\lambda_j'},$$

$$\sum_1^k l_j D_{\lambda_j'} = \frac{m-p!}{m!} D_\lambda \left(- \frac{1}{p} \sum_1^k l_j A_j \right)$$

where

$$\sum_1^k l_j A_j = \left[x(x-1) \cdots (x-p+1) \frac{f(x-p)}{f(x)} \right]_{1/x^2}$$

(that is, the coefficient of $1/x^2$ in the expansion of $x(x-1) \cdots (x-p+1) f(x-p)/f(x)$). Equivalently

$$\sum_1^k l_j A_j = \left[x^2(x-1) \cdots (x-p+1) \frac{f(x-p)}{f(x)} \right]_{1/x}.$$

Putting, as in (1.7), $x = y+k$, we have

$$\left[x^2(x-1) \cdots (x-p+1) \frac{f(x-p)}{f(x)} \right]_{1/x}$$

$$= \left[(y+k) y(y-1) \cdots (y-p+1) \frac{F(y-p)}{F(y)} \right]_{1/y},$$

$$\sum_1^k D_{\lambda_j'} = \left[y(y-1) \cdots (y-p+1) \frac{F(y-p)}{-pF(y)} \right]_{1/y} \frac{m-p!}{m!} D_\lambda.$$

Therefore

$$\sum_1^k (\lambda_j - j) D_{\lambda_j'} = \sum_1^k l_j D_{\lambda_j'} - k \sum_1^k D_{\lambda_j'} \tag{2.3}$$

$$= \left\{\left[(y+k)y\cdots(y-p+1)\frac{F(y-p)}{F(y)}\right]_{1/y}\right.$$

$$\left. - k\left[y\cdots(y-p+1)\frac{F(y-p)}{F(y)}\right]_{1/y}\right\}\frac{m-p!}{m!}D_\lambda \tag{2.3}$$

$$= \left[y^2(y-1)\cdots(y-p+1)\frac{F(y-p)}{-pF(y)}\right]_{1/y}\frac{m-p!}{m!}D_\lambda.$$

From the general expression (2.2) the formulas for $\chi(2, 2)$, $\chi(3, 2)$, $\chi(4, 2)$ are obtained, for example formula for $\chi(2, 2)$ (that is, $p=2$):

By (2.2),

$$\chi(2, 2)_\lambda = \frac{(M_2+2)}{(m-2)(m-3)}\sum_1^k D_{\lambda_j'} - \frac{4}{(m-2)(m-3)}\sum_1^k(\lambda_j - j)D_{\lambda_j'},$$

$$\sum_1^k D_{\lambda_j'} = \chi(2)_\lambda = \frac{M_2 D_\lambda}{m(m-1)},$$

$$\sum_1^k(\lambda_j - j)D_{\lambda_j'} = \frac{D_\lambda}{m(m-1)}\left[y^2(y-1)\frac{F(y-2)}{-2F(y)}\right]_{1/y}.$$

Using (1.3) with $p=2$, multiplying by $y^2(y-1)$, and taking the coefficient of $1/y$, we have

$$\sum_1^k(\lambda_j - j)D_{\lambda_j'} = \frac{D_\lambda(c_4 + 2c_3 + 2m - m^2)}{m(m-1)(m-2)(m-3)}$$

hence

$$\chi(2, 2)_\lambda = \frac{(M_2+2)M_2 - 4(c_4+3c_3+4m-m^2) + 4(c_3+2m)}{m(m-1)(m-2)(m-3)}D_\lambda$$

$$= \frac{M_2^2 - 2M_3 + 4m(m-1)}{m(m-1)(m-2)(m-3)}D_\lambda \tag{2.4}$$

where M_2, M_3 are as defined previously.

3. **The character** $\chi(p, 3)$. Let $\chi(p, 3)$ be the character of the class $\alpha_1 = m-p-3$ unary cycles, $\alpha_3 = 1$ ternary cycle (cycle on 3 letters) and $\alpha_p = 1$ cycle on p letters.

By the recurrence formula (1.2),

$$\chi(p, 3) = \sum_1^k \chi(3)_{\lambda_j'} \tag{3.1}$$

where λ_j', as a subscript, indicates that $\chi(3)$ is the character of a representation $\{\lambda_j'\} = \{\lambda_1, \lambda_2, \cdots, (\lambda_j - p), \cdots, \lambda_k\}$ of S_{m-p} corresponding to the class obtained from the original class by dropping the cycle on p-letters; it is, then, the character of a class with one ternary cycle and all other cycles unary.

$\chi(3)_{\lambda_j'}$ is known, it is given by §1, cf. (5.2)

$$\chi(3)_{\lambda_j'} = [1/2(M_3)_j' - 3/2(m-p)(m-p-1)]\frac{m-p-3!}{m-p!}D_{\lambda_j'}$$

where $(M_3)_j'$ is the same as M_3 but has $\lambda_j - p$ in place of λ_j, m is replaced by $m-p$ as $\{\lambda_j'\}$ is a representation of S_{m-p}, and $D_{\lambda_j'}$ is the dimension of $\{\lambda_j'\}$.

$$M_3 = \sum_1^k [2(\lambda_j - j)^3 + 3(\lambda_j - j)^2 + (\lambda_j - j) + 2j^3 - 3j^2 + j]$$

is a function of $\lambda_1, \lambda_2, \cdots, \lambda_k$. $(M_3)_j'$ is found by substituting $\lambda_j - p$ for λ_j,

$$(M_3)_j' = M_3 - 6p(\lambda_j - j)^2 + 6p(p-1)(\lambda_j - j) - p^2(2p-3).$$

$$\chi(3)_{\lambda_j'} = [1/2M_3 - 3p(\lambda_j - j)^2 + 3p(p-1)(\lambda_j - j) - p^2/2(2p-3) - 3/2(m-p)(m-p-1)]D_{\lambda_j'}\frac{(m-p-3)!}{m-p!}$$

and by (3.1)

$$\chi(p,3)_\lambda = \sum_1^k \left[\frac{1}{2}M_3 - 3p(\lambda_j - j)^2 + 3p(p-1)(\lambda_j - j) - \frac{p^2}{2}(2p-3) - \frac{3}{2}(m-p)(m-p-1)\right]D_{\lambda_j'}\frac{(m-p-3)!}{m-p!}$$

$$= \left[\frac{1}{2}M_3 - p^2/2(2p-3) - \frac{3}{2}(m-p)(m-p-1)\right]\frac{(m-p-3)!}{m-p!}\sum_1^k D_{\lambda_j'} - 3p\sum_1^k [(\lambda_j - j)^2 - (p-1)(\lambda_j - 1)]\frac{m-p-3!}{m-p!}D_{\lambda_j'}. \tag{3.2}$$

$\sum_{j=1}^k D_{\lambda_j'} = \chi(p)$ and is known by the results of §1.

$$\sum_1^k [(\lambda_i - j)^2 - (p-1)(\lambda_i - j)]D_{\lambda'_j}$$

$$= \sum_1^k [(l_i - k)^2 - (p-1)(l_i - k)]D_{\lambda'_j}$$

where $\lambda_j - j = l_j - k$ as previously in §1.

$$\sum_1^k (l_j - k)D_{\lambda'_j}$$

$$= \left[y^2(y-1)\cdots(y-h+1)\frac{F(y-h)}{-pF(y)} \right]_{1/y} \frac{m-p!}{m!} D_\lambda, \tag{3.3}$$

$$\sum_1^k l_i^2 D_{\lambda'_j} = \frac{m-p!}{m!} D_\lambda \left(-1/p \sum_1^k l_i^2 A_i \right),$$

as in §1, where $A_j = l_j(l_j-1)\cdots(l_j-p+1)f(l_j-p)/f'(l_j)$ and $f(x) = \prod_1^k (x-l_j)$ and

$$\sum_1^k l_i^2 A_i = \left[x(x-1)\cdots(x-p+1)\frac{f(x-p)}{f(x)} \right]_{1/x^2}$$

$$= \left[x^2(x-1)\cdots(x-p+1)\frac{f(x-p)}{f(x)} \right]_{1/x}$$

$$= \left[(y+k)^2 y(y-1)\cdots(y-p+1)\frac{F(y-p)}{F(y)} \right]_{1/y},$$

$$\sum_1^k (l_i - k)^2 A_i = \left[\{(y+k)^2 - 2k(y+k) + k^2\} y(y-1) \right.$$

$$\left. \cdot(y-p+1)\frac{F(y-p)}{F(y)} \right]_{1/y}$$

$$= \left[y^3(y-1)\cdots(y-p+1)\frac{F(y-p)}{F(y)} \right]_{1/y},$$

$$\sum_1^k (\lambda_i - j)^2 D_{\lambda'_j} = \left[y^3(y-1)\cdots(y-p+1)\frac{F(y-p)}{-pF(y)} \right]_{1/y}$$

$$\cdot \frac{m-p!}{m!} D_\lambda.$$

From these expressions, $\chi(p, 3)_\lambda$ is obtained. (If $p=2$ the character $\chi(2, 3)$ is obtained, it is of course the same expression as in §2.)

Taking $p=3$, we have

$$\chi(3,3)_\lambda = [1/2M_3 - 27/2 - 3/2(m-3)(m-4)]\frac{m-6!}{m-3!}\sum_1^k D_{\lambda_j'} - 9\sum_1^k [(\lambda_j - j)^2 - 2(\lambda_j - j)]D_{\lambda_j'}\frac{m-6!}{m-3!},$$

$$\sum_1^k D_{\lambda_j'} = \chi(3) = \frac{[1/2M_3 - 3/2m(m-1)]D_\lambda}{m(m-1)(m-2)},$$

$$\sum_1^k [(\lambda_j - j)^2 - 2(\lambda_j - j)]D_{\lambda_j'} = \frac{D_\lambda}{m(m-1)(m-2)}\left[y^2(y-1)(y-2)^2\frac{F(y-3)}{-3F(y)}\right]_{1/y}.$$

Using the expansion of $F(y-3)/-3F(y)$ given by (1.8) with $p=3$, multiplied by $y^2(y-1)(y-2)^2$, and collecting the coefficient of $1/y$, finally we obtain the expression for $\chi(3, 3)$,

$$\chi(3,3) = [1/4M_3^2 - 18M_5 + 54M_3^2 - 6(m^2 - 13m - 39)M_3 + 9m(m-1)(m^2 - 13m + 34)]D_\lambda\frac{m-6!}{m!},$$

with M_2, M_3, M_5 as defined previously.

4. **The character $\chi(2, 2, 2)_\lambda$.** It is evident that, by means of the recurrence formula, formulas can be found for characters of classes consisting of three non-unary cycles. The expressions involved, however, become too complicated for easy computation. To illustrate the method it is sufficient to take the simplest case, the class consisting of three binary cycles and $m-6$ unary cycles.

By the recurrence formula (1.2),

$$\chi(2,2,2)_\lambda = \sum_{j=1}^k \chi(22)_{\lambda_j'},$$

where $\chi(2, 2, 2)_\lambda$ is the required character of the representation $\{\lambda\}$ of S_m, $\chi(2, 2)_{\lambda_j'}$ is the character of the class, consisting of 2 binary cycles and $m-6$ unary cycles, of S_{m-2}, and $\{\lambda_j'\}$ is the representation $\{\lambda_1, \lambda_2, \cdots, \lambda_j - 2, \cdots, \lambda_k\}$ of S_{m-2}. $\chi(2, 2)_{\lambda_j'}$ is known, it is given by (2.3) or (5.5).

$$\chi(2,2)_{\lambda_j'} = \sum_{i=1}^{k} \frac{[(M_2)_j']^2 - 2(M_3)_j' + 4(m-2)(m-3)}{(m-2)(m-3)(m-4)(m-5)}.$$

$(M_2)_j'$ is M_2 written for λ_j', that is, $\lambda_j - 2$ is written for λ_j, as $M_2 = \sum_1^k \lambda_j(\lambda_j - 2j + 1)$.

$$[(M_2)_j']^2 = M_2^2 - 8(\lambda_j - j)M_2 + 4M_2 + 16(\lambda_j - j)^2 - 16(\lambda_j - j) + 4,$$

$$(M_3)_j' = M_3 - 12(\lambda_j - j)^2 + 12(\lambda_j - j) - 6,$$

$$\chi(2,2,2) = \sum_1^k [M_2^2 + 4M_2 - 2M_3 + 4m^2 - 20m + 40]D_{\lambda_j'} \frac{m-6!}{m-2!} + \sum_1^k [40(\lambda_j - j)^2 - 8(M_2 + 5)(\lambda_j - j)D_{\lambda_j'}] \frac{m-6!}{m-2!}. \tag{4.1}$$

The first of these two sums is $[M_2^2 + 4M_2 - 2M_3 + 4m^2 - 20m + 40]M_2 D_\lambda (m-6)!/m!$. The second is obtained by the same reasoning as in the previous sections,

$$\sum_1^k [40(\lambda_j - j)^2 - 8(M_2 + 5)(\lambda_j - j)]D_{\lambda_j'} = \left[(40y^2 - 8(M_2 + 5)y)y(y-1)\frac{F(y-2)}{-2F(y)}\right]_{1/y} D_\lambda \cdot \frac{m-6!}{m!} = \left[8y(y - M_2 - 5)y(y-1)\frac{F(y-2)}{-2F(y)}\right]_{1/y} D_\lambda \cdot \frac{m-6!}{m!}.$$

The expansion of $F(y-2)/-2F(y)$ is found from (1.8), and taking the coefficient of $1/y$ when this series is multiplied by $8y^2(y-1)\cdot(y-M_2-5)$, we get

$$\chi(2,2,2)_\lambda = [M_2^3 - 6M_2M_3 + 40M_4 + 12(m^2 - 9m + 10)M_2]D_\lambda \frac{m-6!}{m!}.$$

5. Results.

$$\frac{\chi(2)_\lambda}{D_\lambda} \frac{m!}{m-2!} = M_2 \qquad \text{(Frobenius)}, \tag{5.1}$$

$$\frac{\chi(3)_\lambda}{D_\lambda} \frac{m!}{m-3!} = \frac{M_3}{2} - 3/2m(m-1) \qquad \text{(Frobenius)}, \tag{5.2}$$

$$\frac{\chi(4)_\lambda}{D_\lambda} \frac{m!}{m-4!} = M_4 - 2(2m-3)M_2 \qquad \text{(Frobenius)}, \tag{5.3}$$

$$(5.4)\qquad \frac{\chi(5)_\lambda}{D_\lambda}\frac{m!}{m-5!} = \frac{M_5}{2} + \frac{49-15m}{6}M_3 - 5/2M_2^2 + 5/6m(m-1)(5m-19),$$

$$(5.5)\qquad \frac{\chi(2,2)_\lambda}{D_\lambda}\frac{m!}{m-4!} = M_2^2 - 2M_3 + 4m(m-1),$$

$$\frac{\chi(3,2)_\lambda}{D_\lambda}\frac{m!}{m-5!} = \frac{1}{2}[M_2M_3 - 6M_4 - 3/2(m^2-13m+16)M_2],$$

$$\frac{\chi(4,2)_\lambda}{D_\lambda}\frac{m!}{m-6!} = M_2M_4 - 4M_5 - 2(2m-11)M_2^2 + 4/3(12m-34)M_3 - 32/3m(m-1)(2m-7),$$

$$\frac{\chi(3,3)_\lambda}{D_\lambda}\frac{m!}{m-6!} = \frac{1}{4}[M_3^2 - 18M_5 + 54M_2^2 - 6(m^2-13m-39)M_3 + 9m(m-1)(m^2-13m+34)],$$

$$\frac{\chi(2,2,2)}{D_\lambda}\frac{m!}{m-6!} = M_2^3 - 6M_2M_3 + 40M_4 + 12(m^2-9m+10)M_2$$

where

$$M_3 = \sum_1^a [b_i(b_i+1) - a_i(a_i+1)]$$

$$= \sum_1^b [(\lambda_j - j)(\lambda_j - j + 1) - j(j-1)],$$

$$M_3 = \sum_1^a [b_i(b_i+1)(2b_i+1) + a_i(a_i+1)(2a_i+1)]$$

$$= \sum_1^b [(\lambda_j - j)(\lambda_j - j + 1)(2\lambda_j - 2j + 1) + j(j-1)(2j-1)],$$

$$M_4 = \sum_1^a [b_i^2(b_i+1)^2 - a_i^2(a_i+1)^2]$$

$$= \sum_1^b [(\lambda_j - j)^2(\lambda_j - j + 1)^2 - j^2(j-1)^2],$$

$$M_s = \sum_1^e [b_i^2(b_i+1)^2(2b_i+1) + a_i^2(a_i+1)^2(2a_i+1)]$$

$$= \sum_1^k [(\lambda_i - j)^2(\lambda_i - j + 1)^2(2\lambda_i - 2j + 1) + j^2(j-1)^2(2j-1)],$$

$$D_\lambda = \frac{m!\Delta(l)}{l_1!l_2!\cdots l_k!} = \frac{m!\Delta(b)\Delta(a)}{\prod_1^e b_i! \prod_1^e a_i! \prod_{i,q=1}^e (b_i + a_q + 1)}.$$

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A PROOF OF A CONJECTURE OF VANDIVER

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The Wedderburn theorem that every finite division ring is commutative has been extended by several authors [1].[1] Vandiver, in his paper *The p-adic representation of rings* [2] conjectured the following generalization of Wedderburn's theorem: every finite, non-commutative ring contains an element which is a divisor of zero and is not in the centrum.

In this paper we give a short and simple proof of this conjecture. We also exhibit one generalization of it which was pointed out to us by the referee.

THEOREM. *A finite non-commutative ring contains an element which is a divisor of zero and is not in the centrum of the ring.*

PROOF. Let R be the ring, C its centrum and N its radical. If every element of R is a divisor of zero, then the theorem is trivially true. So we may assume that D, the set of elements in R which are not divisors of zero, is not empty. We claim that D is a multiplicative group. For if $x, y, z \in D$, then $xy \in D$; and if $xy = zy$ or $yx = yz$, then $x = z$. Hence, since D is finite, it forms a group under multiplication, and so contains a two-sided unit element, 1. We assert that 1 is a unit element for R. For if $r \in R$, then it can be written as $r = xd$ where $d \in D$ (since all the left multiples of d are distinct and so cover all of R). Thus $r1 = (xd)1 = x(d1) = xd = r$. Similarly 1 is a left unit.

Now if R contains a nonzero idempotent $e \neq 1$, then for all $x \in R$, $xe(1-e) = x(1-e)e = 0$, and so if the theorem were false, both xe and $x(1-e)$ would be in C; hence $xe + x(1-e) = x$ would be in C. And thus $R = C$, contradicting that R is non-commutative. Hence we may assume that 1 is the only nonzero idempotent in R.

Since $1 \notin N$, $N \neq R$. Let $x \notin N$. Then Rx is a non-nilpotent left ideal, and so contains a nonzero idempotent [3]; thus Rx contains 1. That is, if $X \notin N$, then x is regular. Hence $R - N$ is a finite division ring, which by the Wedderburn theorem is commutative. But the multiplicative group of a finite field is cyclic; that is, there exists an $\bar{a} \in R - N$ such that for every $\bar{x} \neq 0$ in $R - N$, there exists an integer s so that $\bar{x} = \bar{a}^s$. Let $a \in R$ map an $\bar{a} \in R - N$. Whence for every $x \in R$, $x \notin N$ we can find an integer S so that $a^S - x \in N$. If N is not contained

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

in C, then there exists an element $b \in N$, $b \notin C$ such that $b^n = 0$, in which case the theorem would be true. So we may assume that $N \subset C$. Since for every $x \in R$, $x \notin N$ we can find an S so that $a^s - x \in N \subset C$, $(a^s - x)a = a(a^s - x)$; thus $ax = xa$. Also if $x \in N$, since $N \subset C$, $ax = xa$. Thus $a \in C$. Since R is non-commutative, we can find an $x \notin C$. Thus, since $x \notin N$, for some integer s, $a^s - x \in N \subset C$. And so, since $a \in C$, $a^s \in C$, we conclude that $x \in C$. This is a contradiction; and so we have the theorem.

We are greatly indebted to the referee for pointing out to us the following generalization of the theorem proved above.

THEOREM. *Let R be a ring with the property that every element generates a finite subring. Then R is commutative if all the divisors of zero of R are in the centrum.*

PROOF. Without loss of generality we may assume that every element has characteristic some power of a fixed prime p. Let A be the set of divisors of zero of R and C the centrum; by assumption $A \subset C$. If $x \in R$, $x \notin A$, then $x = x^{r+1}$ for some integer $r > 0$; and as $x(x^r b - b) = (bx^r - b)x = 0$ for all $b \in R$, $x^r = 1$, the identity of R. If $A = (0)$ the theorem follows as a corollary of a theorem of Jacobson [**4**, Theorem 11, p. 702].

Let us now assume that $A \neq (0)$. If $a \neq 0$ is in A, $c \in R$, then $ca \in A$. Since $A \subset C$, $(bc - cb)a = 0$ for all b, $c \in R$, $a \in A$. Thus $bc - cb \in A$, and so $(bc - cb)^2 = 0$. For any $b \in R$, $pb \in A$, so that $p(bc - cb) = 0$ for all b, $c \in R$. If $bc - cb = e$, then $bc = cb + e$, $e \in A$, $pe = 0$, $e^2 = 0$ and $(bc)^p = (cb + e)^p = (cb)^p$. If for b, $c \in R$, $bc \in A$, then $bc = cb$. If $bc \notin A$, then $(bc)^r = 1$ for some $r > 0$. Let $r = np^k$, $(n, p) = 1$. Then $((bc)^n - 1)^q = 0$ for q some power of p. Thus $(bc)^n - 1 \in A$, and so $(bc)^n \in C$. Hence $b(bc)^n = (bc)^n b$, and so $(bc)^n = (cb)^n$. As $sn + tp = 1$, $bc = (bc)^{sn}(bc)^{tp} = (cb)^{sn}(cb)^{tp} = cb$, and the theorem follows.

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ON EUCLIDEAN LOCAL GROUPS SATISFYING CERTAIN CONDITIONS

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Let G be a euclidean local group, that is, a topological local group the space of which is homeomorphic to a euclidean space. The purpose of this note is to prove the following theorem:

THEOREM. *Let G have a neighborhood U of identity e, in which a metric $\rho(x, y)$ can be introduced satisfying the following conditions:*

(A) *If $x, y, xy \in U$,[1] then $K_2\rho(y, e) \leqq \rho(xy, x) \leqq K_1\rho(y, e)$.*

(B) *If $x, x^2, \cdots, x^{2^n}, x^{2^n} \in U$, $y, y^2, \cdots, y^{2^n}, y^{2^n} \in U$, then $K_4 2^n \rho(x, y) \leqq \rho(x^{2^n}, y^{2^n}) \leqq K_3 2^n \rho(x, y)$, with positive constants K_i, $i = 1, 2, 3, 4$.*

Then G is a local Lie group, and vice versa.

P. A. Smith [2][2] has obtained a necessary and sufficient condition for G to be a local Lie group. He says that if we can introduce into a neighborhood of e a coordinate system $a^1, \cdots, a^r$, with respect to which the product function ab is expressible in the form, written vectorially,

$$ab = a + b + |a| F(a, b)$$

where $|a| = (\sum (a^i)^2)^{1/2}$ and where F satisfies the sole condition that $F \to 0$ as $a \to e$, $b \to e$, then G is a local Lie group and vice versa. A coordinate system satisfying this condition is called by him (right) regular. It is shown [1; 2] that a coordinate system in which the product function ab is of class C^1 with respect to b fixing a is regular and that the euclidean metric of a regular coordinate system satisfies our conditions (A) and (B).

Our assumptions (A) and (B) are mainly used to select a uniformly convergent subsequence from the function family $P_n(x, y) = (x^{1/2^n} \cdot y^{1/2^n})^{2^n}$, $n = 1, 2, \cdots$, which, in some sense, corresponds to differentiating the product function at the identity e.

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LEMMA 1. *Let U be a neighborhood of e in G which contains no sub-*

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[1] In details, we must write—instead of $xy \in U$—xy exists and is contained in U. But we shall often omit "exists" in such a case in the following for simplicity.

[2] Numbers in brackets refer to the references cited at the end of the paper.

group in the large except $\{e\}$. *Then there exists a neighborhood* V *of* e *such that* $x, y, x^2=y^2\in V$ *implies* $x=y$.

PROOF.[3] We can assume without loss of generality that $\overline{U}$ is compact and for any arbitrary three elements x, y, z of U the products $(xy)z$ and $x(yz)$ exist. Take W such that $U\supset W^4$, $W=W^{-1}$. As $\overline{U}$ is compact, there exists a neighborhood V of e such that for any $g\in\overline{U}$, we have $g^{-1}Vg\subset W$. Now if $x, y, x^2=y^2\in V$, put $x^{-1}y=a$. Then $axa=x$, $a\in V^2\subset W\subset U$, and it follows that if $a, a^2, \cdots, a^m\in U$, then $a^{2m}=x^{-1}a^{-m}xa^m\in x^{-1}W\subset W^2\subset U$, $a^{2m+1}=a^{2m}\cdot a\in W^2W^2\subset U$. This means that for any integer n, $a^n\in U$; consequently from the assumption $a=e$, $x=y$.

REMARK. In Lemma 1 the assumption that G is euclidean is not necessary. Let σ be a mapping $y\to y^2$. If G is euclidean and if U is sufficiently small, then $\sigma(U)$ is an open subset of G and σ is a homeomorphism between U and $\sigma(U)$. The proof is clear from the Brouwer's theorem on the invariance of domain and from the fact that $\overline{U}$ is compact and that σ is one-to-one.

LEMMA 2. *Let* G *have a neighborhood* U *of* e *such that for any element* y *of* $U-\{e\}$ *there exists an integer* n *satisfying* $y^{2^n}\notin\overline{U}$. *Then, if* V *is a sufficiently small neighborhood of* e, *we can construct a real-valued continuous function* $f(y)$ *defined on* V *satisfying the following conditions*:

(i) *If* $y, y^2\in V$, *then* $f(y^2)\geqq f(y)$.

(ii) $f(y)=0$ *if and only if* $y=e$ *provided* $y\in V$.

PROOF. *If* $y, y^2, y^{2^2}, \cdots, y^{2^n}\in\overline{V}$, $y^{2^{n+1}}\in\overline{V}$, we put $n=\delta_V(y)$. Take an element p of V, we note that if $k\leqq\delta_V(p)$, then $p\neq p^{2^k}$. Consequently if we take a sufficiently small neighborhood W of e, then for any $i<j\leqq\delta_V(p)$, we have $\sigma^i(Wp)\cap\sigma^j(Wp)=\Theta$, $\sigma^{\delta_V(p)+1}(Wp)\cap\overline{V}=\Theta$, where $\sigma^0(y)=y$, $\sigma^2(y)=y^2$, $\sigma^{i+1}(y)=\sigma^1(\sigma^i(y))$.

Construct a continuous function $f_p^0(y)$ defined on V such that

$$0\leqq f_p^0(y)\leqq 1;\ \text{if } y\notin Wp,\ \text{then } f_p^0(y)=0;\ \text{and } f_p^0(p)=1.$$

For any integer $k\leqq\delta_V(p)$ put

$$\text{if } y\notin\sigma^k(Wp),\qquad f_p^k(y)=0,$$
$$\text{if } y\in\sigma^k(Wp),\qquad f_p^k(y)=f_p^0(y^{1/2^k}),\ \text{where } (y^{1/2^k})^{2^k}=y.$$

If V is sufficiently small, by the remark to Lemma 1, $f_p^k(y)$ is a continuous function defined on V. Put

[3] This simple proof was suggested to the author by T. Hayashida. The author's original proof is more complicated.

$$f_p(y) = \sum_{k=0}^{h_V(p)} f_p^k(y).$$

It is easy to see that $f_p(y)$ is a continuous function defined on V and $0 \leqq f_p(y) \leqq 1$, $f_p(p) = 1$, $f_p(y^2) \geqq f_p(y)$. As $\bigcup_{p \in -V[e]} V_p = V - \{e\}$, where $V_p = \{y;\ f_p(y) > 0\}$, we can select a countable set $\{p_n\}$ such that $\bigcap_{n=1}^{\infty} V_{p_n} = V - \{e\}$. Put

$$f(y) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_{p_n}(y).$$

It is clear that $f(y)$ satisfies our conditions (i) and (ii).

LEMMA 3. *Under the same assumption as in Lemma 2, there exists a neighborhood W of e such that for an arbitrary element x of W, W contains a unique element $x^{1/2}$ satisfying $(x^{1/2})^2 = x$.*

PROOF. Take a sufficiently small neighborhood V of e, where we can construct the function $f(y)$ as stated in Lemma 2. Put $V_\epsilon = \{y; f(y) < \epsilon\}$. If V is sufficiently small, by the Remark to Lemma 1, $\sigma(V)$ is an open set. From the condition (ii) which $f(y)$ satisfies, there exists $\epsilon > 0$, $V_\epsilon \subset \sigma(V) \cap V$. This means that for any element y of V_ϵ there exists $y^{1/2} \in V$ such that $(y^{1/2})^2 = y$. But since $f(y^{1/2}) \leqq f(y) < \epsilon$, $y^{1/2} \in V_\epsilon$. The uniqueness of $y^{1/2}$ has already been proved in Lemma 1.

By Lemma 3, there exists $x^{1/2^n}$ such that $(x^{1/2^n})^{2^n} = x$ and $x^{1/2^n} \in W$ for any element x of W. We can assume without loss of generality $W \subset U$, where U is a neighborhood of e satisfying the assumption of our theorem. We shall write $|x|$ instead of $\rho(x, e)$. Then by (B), $|x^{1/2^n}| \leqq (1/K_4)(1/2^n)|x|$. We can prove, by induction, the existence of $(x^{1/2^n})^m$ and

$$(*) \qquad |(x^{1/2^n})^m| \leqq \frac{K_1}{K_4} \frac{m}{2^n} |x|$$

for any arbitrary integer $m \leqq 2^n$. Assume, in fact, $|(x^{1/2^n})^m| \leqq (K_1/K_4)(m/2^n)|x|$ for some $m < 2^n$, then

$$\begin{aligned} |(x^{1/2^n})^{m+1}| &= |(x^{1/2^n})^m x^{1/2^n}| \leqq K_1 |x^{1/2^n}| + |(x^{1/2^n})^m| \\ &\leqq \frac{K_1}{K_4} \frac{1}{2^n} |x| + \frac{K_1}{K_4} \frac{m}{2^n} |x| \\ &= \frac{K_1}{K_4} \frac{m+1}{2^n} |x|, \end{aligned}$$

as we can see easily from (A) that

(C) $$|xy| \leqq K_1|y| + |x|.$$

By using the uniqueness of $x^{1/2}$, it is easy to see that $m/2^n = m'/2^{n'}$ implies $(x^{1/2^n})^m = (x^{1/2^{n'}})^{m'}$. Now we put $f(m/2^n) = (x^{1/2^n})^m$; then $f(m/2^n)f(m'/2^{n'}) = f(m/2^n + m'/2^{n'})$, and from these and from the inequality (*) follow the uniform continuity of the function f on the set $\{m/2^n;\ m \leqq 2^n\}$, which is everywhere dense in the interval $[0, 1]$. Therefore f can be extended to a continuous function $\bar{f}$ defined on $[0, 1]$. If we put $x^\lambda = f(\lambda)$, $x^{-\lambda} = (x^{-1})^\lambda$, for any $0 \leqq \lambda \leqq 1$, then $|x^\lambda| \leqq (K_1/K_4)|x|$ and $x^\lambda \cdot x^\mu = x^{\lambda+\mu}$ if both sides have meaning. This proves the following lemma.

LEMMA 4. *Under the same assumption as in the theorem, there exist neighborhoods W_1 and W_2 of e such that for any element x of W_1, there exists a unique one-parameter subgroup x^λ contained in W_2.*

From (A),

(D') $$\rho(ax, ay) \leqq K_1\rho((ay)^{-1} \cdot ax, e) = K_1\rho(y^{-1} \cdot x, e) \leqq \frac{K_1}{K_2}\rho(x, y).$$

Take a sufficiently small neighborhood V of e, and put

$$\Sigma = \{x;\ x \in V,\ X^2 \notin V\}, \qquad C = \min_{x \in \Sigma} |x|,$$

$$\Omega = \{w;\ w = a^{-1}xa,\ x \in \Sigma,\ a \in V\}, \qquad D = \max_{x \in \Omega} |x|.$$

Then $\bar{\Sigma} \oplus e$, $C > 0$, $D > 0$, and

(**) $$|a^{-1}xa| \leqq \frac{D}{C}|x| \qquad \text{for } x \in \Sigma,\ a \in V.$$

Take $y \in V$, then for some integer n, $y^{2^n} \in \Sigma$ and for every integer $0 \leqq m \leqq n$, $y^{2^m} \in V$. From (B) and (**)

$$|a^{-1}ya| \leqq \frac{1}{2^n}\frac{1}{K_4}|a^{-1}y^{2^n}a| \leqq \frac{1}{2^n}\frac{1}{K_4}\frac{D}{C}|y^{2^n}| \leqq \frac{K_3D}{K_4C}|y|,$$

(D'') $$\rho(xa, ya) \leqq K_1\rho(a^{-1}y^{-1}xa, e) \leqq \frac{K_1K_3D}{K_4C}\rho(y^{-1}x, e)$$

$$\leqq \frac{K_1K_3D}{K_2K_4C}\rho(x, y),$$

if x, y, a is sufficiently near to e.

From (D') and (D'') we can see that for a sufficiently small neigh-

borhood V of e, there exists $A>0$ such that

(D)[4] if $x, y, x', y' \in \overline{V}$, $\rho(xy, x'y') \leqq A\{\rho(x, x') + \rho(y, y')\}$.

If we take a sufficiently small neighborhood V of e, from (C), (B), and (D) we can easily prove the existence of $P_n(x, y) = (x^{1/2^n} y^{1/2^n})^{2^n}$ and

$$\text{(E)} \quad |P_n(x, y)| = |(x^{1/2^n} y^{1/2^n})^{2^n}| \leqq \frac{K_1 K_2}{K_4}(|x| + |y|),$$

$$\text{(F)} \quad \rho(P_n(x, y), P_n(x', y')) \leqq \frac{K_2 A}{K_4}\{\rho(x, x') + \rho(y, y')\}$$

for arbitrary $x, y, x', y' \in \overline{V} \subset U$.

Therefore $P_n(x, y)$, $n=0, 1, 2, \cdots$, is an equi-continuous and uniformly bounded family of functions defined on $\overline{V} \times \overline{V}$, and consequently a uniformly convergent subsequence $\{P_{n'}(x, y)\}$ can be selected from $\{P_n(x, y)\}$. Put $P(x, y) = \lim_{n' \to \infty} P_{n'}(x, y)$.

Now we shall prove that if we define the product $x \circ y = P(x, y)$, V becomes an abelian local Lie group. We denote this local group by H.

(1) Commutativity is clear from $(x^{1/2^n} y^{1/2^n})^{2^n} = x^{1/2^n}(y^{1/2^n} x^{1/2^n})^{2^n} (x^{1/2^n})^{-1}$.

(2) Associative law: Put $P(x, y) = \epsilon_{n'} P_{n'}(x, y)$, then $\lim_{n' \to \infty} \epsilon_{n'} = e$. Using (D), (E), and (F),

$$\rho(P_{n'}(P(x, y), w), P_{n'}(P_{n'}(x, y), w)) \leqq \frac{K_2 A^2}{K_4} \rho(\epsilon_{n'}, e),$$

from the definition $P_{n'}(P_{n'}(x, y), w) = P_{n'}(x, P_{n'}(y, w))$. From these we can see $(x \circ y) \circ w = x \circ (y \circ w)$.

(3) $x^\lambda \circ x^\mu = x^{\lambda+\mu}$, which shows the existence of the inverse, and that every one-parameter subgroup of G is also that of H.

(4) As H is abelian, we can find easily $x_1, x_2, \cdots, x_r$ of H (where r is the dimension of G) such that $W = \{y;\ y = x_1^{\lambda_1} \circ x_2^{\lambda_2} \circ \cdots \circ x_r^{\lambda_r}, |\lambda_i| \leqq 1,\ i=1, 2, \cdots, r\}$ is a neighborhood of e and mapping $(\lambda_1, \lambda_2, \cdots, \lambda_r) \to x_1^{\lambda_1} \circ x_2^{\lambda_2} \circ \cdots \circ x_r^{\lambda_r}$ is topological. Moreover

$$(x_1^{\lambda_1} \circ \cdots \circ x_r^{\lambda_r}) \circ (x_1^{\lambda'_1} \circ \cdots \circ x_r^{\lambda'_r}) = (x_1^{\lambda_1+\lambda'_1} \circ \cdots \circ x_r^{\lambda_r+\lambda'_r}).$$

Thus we can see that H is a Lie group.

LEMMA 5. *Under the same assumptions as in the theorem the local group consisting of inner automorphisms of G is a linear group.*

PROOF. From the uniqueness of $x^{1/2}$, it is easy to see $px^{1/2}p^{-1}$

[4] The proof of this inequality is essentially due to P. A. Smith [2].

$=(pxp^{-1})^{1/2}$, for arbitrary $x, y, p \in V$. From this and from the sufficient smallness of $|P_n(x, y)|$, we can deduce immediately that for arbitrary $p, x, y \in V$, $P_n(p^{-1}xp, p^{-1}yp) = p^{-1} \cdot P_n(x, y)p$, which implies $(p^{-1}xp) \circ (p^{-1}yp) = p^{-1}(x \circ y)p$. Thus $x \to p^{-1}xp$ is an automorphism of the abelian Lie group H, which proves the lemma.

LEMMA 6.[5] *Let N be a closed local normal subgroup of G, satisfying the following conditions*:

(1) *N is an abelian local Lie group,*

(2) *G/N is a local Lie group,*

(3) *there exists a set M such that for any a sufficiently near to e, Na contains one and only one element a' of M depending continuously on G/N.*

If G contains N satisfying the above conditions, G is a local Lie group.

PROOF. As N is an abelian Lie group, we can introduce a coordinate system $u(\xi^1, \cdots, \xi^n)$ in N satisfying $u(\xi^1, \cdots, \xi^n) \cdot u(\xi'^1, \cdots, \xi'^n) = u(\xi^1+\xi'^1, \cdots, \xi^n+\xi'^n)$. The transformation by an element a of G induces an automorphism in N, given by

$$au\,(\xi^1, \cdots, \xi^n)a^{-1} = u(\xi'^1, \cdots, \xi'^n),$$
$$(\xi'^1, \cdots, \xi'^n) = (\xi^1, \cdots, \xi^n)A_a,$$

where $(A^{a^i_i})$ is a real matrix of degree n. As $G = NM$, we can put for arbitrary elements $p, q \in M$ sufficiently near to e,

$$pq = \Psi(p, q)(p \circ q),$$
$$\Psi(p, q) = u(\Psi^1(p, q), \cdots, \Psi^n(p, q)) \in N, \qquad p \circ q \in M,$$

where, by (3), both $p \circ q$ and $\Psi(p, q)$ are continuous functions on $M \times M$. Then by the product $p \circ q$, M is a local Lie group isomorphic with G/N, and $M \ni p \to A_p$ is a representation of the local Lie group M. Consequently, introducing the canonical coordinate $v(\eta^1, \cdots, \eta^m)$ in M, $A_{V(\eta^1, \cdots, \eta^m)}$ are analytic functions of $(\eta^1, \cdots, \eta^m)$.

From the associative law of the product of G, if p, q, r are sufficiently near to e, we can see easily that

$$(1)\quad \Psi^i(p, q) + \Psi^i(p \circ q, r) = \sum_{j=1}^{n} A^i_{p,j}\Psi^j(q, r) + \Psi^i(p, q \circ r), \quad i = 1, \cdots, n.$$

Let $\phi(p)$ be an arbitrary continuous function defined on M and

[5] In the proof of this lemma, the fact that the space of G is euclidean is not used. To prove our theorem we need only the case when N is the center of G. But we state the lemma in this general form, because this lemma is applicable to some theorem on locally compact groups (cf. [3]).

taking values in N. $M_\phi=\{p_\phi;\ p_\phi=\phi(p)p,\ p\in M\}$ also satisfies the condition (3), if M is replaced by M_ϕ. We can put

$$p_\phi\cdot q_\phi=\Psi_\phi(p,q)(p_\phi\circ q_\phi),$$
$$\Psi_\phi(p,q)\in N,\qquad p_\phi\circ q_\phi\in M_\phi.$$

As $p_\phi\circ q_\phi=(p\circ q)_\phi=\phi(p\circ q)(p\circ q)$, the relation between Ψ^i and Ψ^i_ϕ is given by

$$(2)\quad \Psi^i_\phi(p,q)=\phi^i(p)+\sum_{j=1}^n A^i_{p,j}\phi^j(q)+\Psi^i(p,q)-\phi^i(p\circ q),\ i=1,2,\cdots,n.$$

Take two sufficiently small numbers α and β such that $U=\{v(\eta^1,\cdots,\eta^m);\ |\eta^i|<\alpha\}$ and $V=\{v(\eta^1,\cdots,\eta^m);\ |\eta^i|<\beta\}$ satisfy $U\supset V\circ V\circ V$ and such that if $p,\ q,\ r\in U$, equalities (1) and (2) have meanings. Take a real function $c(p)$ defined on U such that $c(v(\eta^1,\cdots,\eta^m))$ is of class C^∞ with respect to $\eta^1,\cdots,\eta^m$ and which satisfies the following conditions:

If $p\in V$, then $c(p)=0$,

$$\int_V c(v(\eta^1,\cdots,\eta^m))d\eta^1\cdots d\eta^m=\int_V c(v(\eta))d\eta=\int_U c(v(\eta))d\eta=1.$$

Put

$$\phi_0^i(p)=-\int_V\Psi^i(p,v(\eta))c(v(\eta))d\eta=-\int_U\Psi^i(p,v(\eta))c(v(\eta))d\eta$$

and $\phi_0(p)=u(\phi_0^1(p),\cdots,\phi_0^n(p))\in N$. Then from (1) and (2), if $p,\ q\in V$

$$(3)\quad \Psi^i_{\phi_0}(p,q)=\int_V\Psi^i(p,q\circ v(\eta))c(v(\eta))d\eta-\int_V\Psi^i(p,v(\eta))c(v(\eta))d\eta$$
$$=F_1(p,q)-F_2(p),\qquad i=1,2,\cdots,n.$$

Put $q\circ v(\eta^1,\cdots,\eta^m)=v(\eta'^1,\cdots,\eta'^m)$, and changing the integral variable η by η', we can see

$$F_1^i(p,q)=\int_V\Psi^i(p,q\circ v(\eta))c(v(\eta))d\eta$$
$$=\int_{q\circ V}\Psi^i(p,v(\eta'))c(q^{-1}\circ v(\eta'))J(\eta')d\eta'$$
$$=\int_U\Psi^i(p,v(\eta'))c(q^{-1}\circ v(\eta'))J(\eta')d\eta',$$

because $\eta \to \eta'$ is an analytic function of η and $q \circ V \subset U$. As $c(v(\eta'))$, $q \circ r$, and q^{-1} are functions of class C^2, the above equality shows that $F_1^i(p, v(\eta))$ are functions of class C^2 with respect to η when p is fixed. Consequently, by (3), $\Psi_{\phi_0}^i(p, v(\eta))$ are functions of class C^2 with respect to η when p is fixed. From (1), we can get

$$\sum_{j=1}^{n}\left\{\int_U A_{pj}^i c(p)dp\right\}\Psi^j(q, r)$$

$$= \int_U \Psi^i(p, q)c(p)dp + \int_U \Psi^i(p \circ q, r)c(p)dp \tag{4}$$

$$- \int_U \Psi^i(p, q \circ r)c(p)dp.$$

From the fact that if U is sufficiently small, A_p $(p \in U)$ is sufficiently near to the identity matrix and that $\int_U c(p)dp = 1$, it follows that the determinant $\left|\int_U A_{pj}^i c(p)dp\right|$ is not zero. Consequently, using the same argument as above on $\int_U \Psi^i(p \circ q, r)\, c(p)dp$, we can deduce from (4) that if $\Psi^i(p, q)$ is of class C^2 with respect to q when p is fixed, $\Psi^i(p, q)$ is of class C^2 with respect to p when q is fixed. Thus we have proved that $\Psi_{\phi_0}^i(p, q)$ is of class C^2 with respect to both p and q in the coordinate system $v(\eta^1, \cdots, \eta^m)$. As any element p of G is written uniquely as $p = nm_\phi$, $n \in N$, $m_\phi \in M$, we can introduce a coordinate system $w(\zeta^1, \cdots, \zeta^{n+m})$ in G by

$$w(\zeta^1, \cdots, \zeta^{n+m}) = u(\zeta^1, \cdots, \zeta^n)\phi_0(v(\zeta^{n+1}, \cdots, \zeta^{n+m}))v(\zeta^{n+1}, \cdots, \zeta^{n+m}).$$

As $\phi_0(v(\eta)) \in N$ and N is abelian, we can easily verify that the product function $f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m})$ $(i = 1, 2, \cdots, n+m)$ of G with respect to this coordinate system is given by

$$f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m}) = \zeta^i + A_{v(\zeta^{n+1}, \cdots, \zeta^{n+m})j}^i \zeta'^j + \Psi_{\phi_0}^i(v(\zeta^{n+1}, \cdots, \zeta^{n+m}), v(\zeta'^{n+1}, \cdots, \zeta'^{n+m})) \quad (i = 1, 2, \cdots, n),$$

$$f^i(\zeta^1, \cdots, \zeta^{n+m}; \zeta'^1, \cdots, \zeta'^{n+m}) = g^{i-n}(\zeta^{n+1}, \cdots, \zeta^{n+m}; \zeta'^{n+1}, \cdots, \zeta'^{n+m}) \quad (i = n+1, \cdots, n+m),$$

where $g^{i-n}(\zeta^{n+1}, \cdots, \zeta^{n+m}; \zeta'^{n+1}, \cdots, \zeta'^{n+m})$ is the product function of $p \circ q$ with respect to $v(\zeta^{n+1}, \cdots, \zeta^{n+m})$.

Thus, with respect to the coordinate system $w(\zeta)$, the product function of G is of class C^3 and hence G is a local Lie group.

PROOF OF THE THEOREM. When G has the discrete center, Lemma 5 shows that G is a local Lie group.

When G has the non-discrete center N, by Lemma 4, N is an abelian Lie group, and by Lemma 5 again, G/N is a local Lie group. Then we can introduce a canonical coordinate of the second kind by $x_1^{*\lambda_1} \cdots x_m^{*\lambda_m}$, where $x_i^{*\lambda_i}$ $(i=1, 2, \cdots, m)$ are one-parameter subgroups of G/N. Take x_i of G from the coset x_i^* for each i, we can easily show that the set $M=\{y;\ y=x_1^{\lambda_1} \cdots x_m^{\lambda_m},\ |\lambda_i| \leqq 1\}$ satisfies the condition (3) of Lemma 6. Consequently, by Lemma 6, G is a local Lie group. This completes our proof.

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NOTE ON A THEOREM OF KOKSMA

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In 1935 Koksma [2][2] showed, among other things, that the sequence $x, x^2, x^3, \cdots$ is uniformly distributed (mod 1) for almost all $x>1$; that is, that if $N(n, \alpha, \beta, x)$ denotes the number of elements x^j of the sequence $x, x^2, \cdots, x^n$ for which

$$0 \leqq \alpha \leqq x^j - [x^j] < \beta \leqq 1,$$

then

$$\lim_{n\to\infty} \frac{N(n, \alpha, \beta, x)}{n} = \beta - \alpha$$

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[1] The author is indebted to Professor Mark Kac for his help in connection with this paper.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

for almost all $x>1$. The purpose of this note is to provide another proof of this theorem, based on a lemma used in a recent paper [1] of Kac, Salem, and Zygmund on quasi-orthogonal functions. The assertion is contained in the more general theorem:

THEOREM 1. *Let $g(x, n)$ be any function of the real variable x and the positive integral variable n with the following properties in the interval $a<x<b$:*

(i) *dg/dx, d^2g/dx^2 exist,*

(ii) *$g'(x, j)-g'(x, k)$ is monotonic, and is different from zero for $j\neq k$,*

(iii) *for $x=a$ and $x=b$, the inequality*

$$|g'(x, j) - g'(x, k)| \geqq C|j - k|^{\epsilon}$$

is fulfilled for some $C>0$, $0<\epsilon\leqq 1$.

Then the sequence $g(x, 1)$, $g(x, 2)$, $\cdots$ is uniformly distributed (mod 1) *for almost all $x\in(a, b)$.*

This theorem is weaker than Theorem 3 of Koksma's paper, but it is easily verified that the conditions of the theorem are satisfied for

$$g(x, n) = x^n, \quad 1 \leqq a < b,$$

$$g(x, n) = n^t x, \quad \text{every } a, b \quad (t \text{ a positive integer}),$$

$$g(x, n) = n^x, \quad 1 \leqq a < b,$$

$$g(x, n) = x^{M(n)}, \quad 1 \leqq a < b,$$

where $M(n)$ is positive and such that $|M(j)-M(k)|\geqq N$ for some $N>0$ and all $j\neq k$. This last case is Theorem 2 of Koksma's paper.

We use the following specialization of Lemma 1 of [1]: Let $g(x, n)$ $(n=1, 2, \cdots)$ be any sequence of real continuous functions in (a, b), and let m be an integer different from zero. Suppose that for all positive integers j, k with $j\neq k$, we have

$$\left|\int_a^b e^{2\pi i m(g(x,j)-g(x,k))}dx\right| \leqq \frac{C_1}{|j - k|^{\epsilon}}$$

for some ϵ with $0<\epsilon\leqq 1$. Then the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x,n)}}{n^{1-\delta}}$$

converges almost everywhere in (a, b) for every $\delta<\epsilon/2$.

(Lemma 1 of [1] is stated for real-valued functions; in the case of complex-valued functions the integrand should be replaced by $f_j(x)\bar{f}_k(x)$, where $\bar{f}$ is the complex conjugate of f.)

We shall show that for a function g satisfying (i), (ii), and (iii) of the theorem, the conditions of the lemma are also satisfied for each m. The validity of the theorem then follows from Weyl's criterion [3, p. 91] for uniform distribution (mod 1) upon noting that the convergence of $\sum a_n/n$ implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} a_n = 0.$$

Assume (i), (ii), and (iii). Then

$$\left| \int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\} dx \right|$$

$$= \left| \int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\} \cdot \frac{g'(x,j) - g'(x,k)}{g'(x,j) - g'(x,k)} dx \right|$$

$$\leqq \frac{1}{2\pi |m|} \left(\left| \left[\frac{\exp\{2\pi i m(g(x,j) - g(x,k))\}}{g'(x,j) - g'(x,k)} \right]_a^b \right| \right.$$

$$\left. + \left| \int_a^b \exp\{2\pi i m(g(x,j) - g(x,k))\} \frac{d}{dx} (g'(x,j) - g'(x,k))^{-1} dx \right| \right)$$

$$\leqq \frac{1}{2\pi |m|} \left(\frac{1}{|g'(a,j) - (g'a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|} \right.$$

$$\left. + \int_a^b \left| \frac{d}{dx} (g'(x,j) - g'(x,k))^{-1} \right| dx \right).$$

By (ii), this is

$$\leqq \frac{1}{2\pi |m|} \left(\frac{1}{|g'(a,j) - (g'(a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|} \right.$$

$$\left. + \left| \int_a^b \frac{d}{dx} (g'(x,j) - g'(x,k))^{-1} dx \right| \right)$$

$$\leqq \frac{1}{\pi |m|} \left(\frac{1}{|g'(a,j) - g'(a,k)|} + \frac{1}{|g'(b,j) - g'(b,k)|} \right),$$

and this, by (iii), is

$$\leqq \frac{2}{C\pi |m| \cdot |j-k|^{\epsilon}},$$

so that the assumptions of the lemma hold with $C_1=2/C\pi|m|$.

Actually, we have proved considerably more than is stated in Theorem 1, at least for the special functions $g(x, n)$ cited above. In each of these cases, the hypothesis of Theorem 1 holds with $\epsilon=1$. It follows that for every integer $m\neq 0$, the series

$$\sum_{n=1}^{\infty} \frac{e^{2\pi i m g(x,n)}}{n^{1/2+\theta}}, \qquad \theta > 0,$$

converges for almost all $x\in(a, b)$. Using Abel's partial summation formula we deduce the following theorem.

THEOREM 2. *Under the assumptions of Theorem* 1,

$$\sum_{n=1}^{N} e^{2\pi i m g(x,n)} = o(N^{1/2+\theta}), \qquad \theta > 0,$$

for every integer $m\neq 0$ *and for almost all* $x\in(a, b)$.

This is a much stronger statement than an assertion about uniform distribution. In view of its generality it is remarkably close to best possible, since it is known [4] that for $g(x, n)=xn^2$, this sum is not $o(N^{1/2})$ for any irrational x.

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HARVARD UNIVERSITY

ON LEBESGUE'S DENSITY THEOREM

CASPER GOFFMAN

The density theorem of Lebesgue [1] may be stated in the following form: If S is a measurable linear point set, the metric density of S exists and is equal to 0 or 1 almost everywhere. We prove the converse that for every set Z of measure 0 there is a measurable set S whose metric density does not exist at any point of Z. We note, however, that in order for Z to be the set of points for which the metric density of some set S exists but is different from 0 or 1, Z must be both of measure 0 and of first category. As a converse, we show that for every F_σ type set Z of measure 0, thus of first category, there is a measurable set S whose metric density exists but is different from 0 or 1 at every point of Z.

THEOREM 1. *If Z is any set of measure* 0, *there is a measurable set S whose metric density does not exist at any point of Z.*

PROOF. Let $G_1 \supset G_2 \supset \cdots \supset G_n \supset \cdots$ be a sequence of open sets each covering Z. Since Z is of measure 0, the sets G_n may be so chosen that, for every n, the relative measure of G_{n+1} is $1/n$ in each of the disjoint open intervals I_{np}, $p=1, 2, \cdots$, constituting G. Let

$$S = (G_1 - G_2) + (G_3 - G_4) + \cdots + (G_{2k-1} - G_{2k}) + \cdots .$$

S is a measurable set. Let z be any point in Z. For every n, there is a p_n such that z is in I_{np_n}, one of the disjoint open intervals constituting G_n. For odd values of n, the relative measure of S in I_{np_n} is not less than $1-1/n$. For then $S \supset G_n - G_{n+1}$, a set whose relative measure in I_{np_n} is exactly $1-1/n$. For even values of n, $S \subset G_{n+1}$, a set whose relative measure in I_{np_n} is $1/n$. Hence, in this case, the relative measure of S in I_{np_n} is not greater than $1/n$. But the length of I_{np_n} converges to zero, as n increases, both for odd and even values of n. The upper and lower metric densities of S at z are accordingly equal to 1 and 0, respectively, so that the metric density of S does not exist at any point of Z.

Theorem 1 does not assert that the metric density of S exists at every point outside of Z. In fact, if Z is of measure 0 and not measurable Borel there is no set S such that Z is the totality of points at which the metric density of S does not exist. For, the set of points at which the metric density of S does not exist is, as may readily be shown, of type $G_{\delta\sigma}$.

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It is natural to ask whether for every set Z of measure 0 there is a measurable set S whose metric density exists but is different from 0 or 1 at every point of Z. The answer is no, as is shown by the following:

THEOREM 2. *The set of points for which the metric density of a measurable set S exists but is not equal to 0 or 1 is of measure 0 and of first category.*

PROOF. We shall suppose all sets are contained in the open interval $(0, 1)$. Let T be the set of points for which the metric density of S exists, U those points of T for which the metric density of S is 0 or 1, and $Z=T-U$. By the density theorem of Lebesgue, Z is of measure 0 and U is of measure 1. The metric density of S is a function of Baire class 1 on T. For, let x be in T and let $f_n(x)$, $n=1, 2, \cdots$, be the relative measure of S in the interval $(x-1/n, x+1/n)$. For every n, $f_n(x)$ is continuous and is, accordingly, continuous relative to T. Since the metric density of S exists at every point of T, $f(x) = \lim_{n\to\infty} f_n(x)$ exists, is equal to the metric density of S, and is a function of Baire class 1 on T relative to T. Its points of discontinuity must be a set of first category relative to T [2]. On the other hand, U, as a set of measure 1, is everywhere dense in T. Thus, every interval containing a point of Z also contains points of U; that is, points for which $f(x)$ is either 0 or 1. Since for every x in Z, $f(x)$ is different from 0 or 1, Z must be a subset of the set of points of discontinuity of $f(x)$. Z is, accordingly, of first category relative to T and, therefore, relative to $(0, 1)$.

As a partial converse to Theorem 2, we have the following theorem.

THEOREM 3. *If Z is a set of measure 0 which is of type F_σ, there is a measurable set S whose metric density exists and is different from 0 or 1 at every point of Z.*

PROOF. We may again assume that all sets are contained in the interval $(0, 1)$. $Z=\sum_{k=1}^{\infty} Z_k$ where the Z_k are closed sets of measure 0. We associate four sequences of open sets with Z:

$$G_1 = (0, 1) \supset G_2 \supset \cdots \supset G_n \supset \cdots,$$
$$T_1 \supset T_2 \supset \cdots \supset T_n \supset \cdots,$$
$$U_1 \supset U_2 \supset \cdots \supset U_n \supset \cdots,$$
$$V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots;$$

where the G_k and T_k are descending sequences of open sets such that, for every positive integer k,

$$G_k \supset Z - (Z_1 + Z_2 + \cdots + Z_{k-1}),$$
$$T_k = G_k - Z_k,$$
$$G_{k+1} \subset T_k,$$

and U_k, V_k are disjoint subsets of T_k to be defined presently:

T_k, as an open set, consists of a sequence $I_{k1}, I_{k2}, \cdots$ of disjoint open intervals. For every j, there is a positive integer N_j such that

$$\frac{2}{N_j+1} \leqq |I_{kj}| < \frac{2}{N_j}.$$

For every $n > N_j+1$, let the points in I_{kj} whose distance from an end point of I_{kj} is less than $1/n$ and greater than $2^{-1}(1/(n+1)+1/n)$ belong to $U_k^{(n)}$ and those whose distance from an end point of I_{kj} is less than $2^{-1}(1/(n+1)+1/n)$ and greater than $1/(n+1)$ belong to $V_k^{(n)}$. Divide the open interval of points in I_{kj} whose distance from an end point of I_{kj} exceeds $1/(N_j+1)$ into two open intervals of equal length, letting one belong to $U_k^{N_j}$ and the other to $V_k^{N_j}$. Let

$$U_k = \sum_{\nu=1}^{\infty} U_k^{(\nu)}, \qquad V_k = \sum_{\nu=1}^{\infty} V_k^{(\nu)}.$$

Then, for every k,

$$T_k = U_k + V_k + D_k$$

where D_k is a finite or denumerable set consisting of the end points of the disjoint open intervals of U_k, V_k. Moreover, we restrict the choice of the sets G_k, as we may since Z is of measure 0, so that for every $l = 1, 2, \cdots, k-1$, and for every n, the relative measure of G_k in each of the disjoint open intervals of the sets $U_{k-l}^{(n)}$, $V_{k-l}^{(n)}$ does not exceed $1/n^l$. We now define S by the expression

$$(1) \quad S = (U_1 - G_2) + (U_2 - G_3) + \cdots + (U_n - G_{n+1}) + \cdots.$$

S is a measurable set. We show that the metric density of S exists and is equal to $1/2$ at every point of Z. For, let $z \in Z$. There is a smallest k such that $z \in Z_k \subset G_k$. Let J be an open interval containing z with $|J|$ less than the distance from z to the complement of G_k. Since G_k is open, $J \subset G_k$ and there is a positive integer n such that

$$(2) \qquad \frac{1}{n+1} \leqq |J| \leqq \frac{1}{n}.$$

Since $z \in Z_k$, $z \notin T_k$. So, if an end point x of J is in one of the disjoint

open intervals of U_k or V_k, the length of this interval is not greater than $2^{-1}(1/n-1/(n+1))=1/2n(n+1)$. For, the distance of x from an end point of the open interval of T_k in which it would be could not exceed $1/n$. The interval J accordingly consists of a set H composed of disjoint open intervals of the sets $U_k^{(\nu)}$, $V_k^{(\nu)}$, $\nu=n, n+1, \cdots$, the same finite number belonging to $U_k^{(\nu)}$ as to $V_k^{(\nu)}$ for every ν; the finite or denumerable set of end points of these intervals; a subset $J\cdot Z_k$ of Z_k; and two remaining open intervals, possibly empty, at the ends of J, the sum of whose lengths is not greater than $1/n(n+1)$. From (2), the relative measure of the sum of these two intervals in J is not greater than $1/n$. Since the relative measures of U_k and V_k in H are evidently each equal to $1/2$, the relative measures of $H\cdot U_k$ and $H\cdot V_k$ in J satisfy

$$(3)\qquad \begin{aligned} \frac{1}{2}-\frac{1}{n} &\leqq \frac{m(H\cdot U_k)}{|J|} \leqq \frac{1}{2}, \\ \frac{1}{2}-\frac{1}{n} &\leqq \frac{m(H\cdot V_k)}{|J|} \leqq \frac{1}{2}. \end{aligned}$$

But the relative measure of G_{k+l}, $l=1, 2, \cdots$, is less than $1/\nu^l$ in each of the disjoint open intervals of $U_k^{(\nu)}$, $V_k^{(\nu)}$, and since H consists of disjoint open intervals of sets $U_k^{(\nu)}$, $V_k^{(\nu)}$, $\nu\leqq n$, the relative measure of G_{k+l}, $l=1, 2, \cdots$, is less than $1/n^l$ in $H\cdot U_k$ and in $H\cdot V_k$. Since $U_{k+l}\subset G_{k+l}$, the relative measure of U_{k+l}, $l=1, 2, \cdots$, is also less than $1/n^l$ in $H\cdot U_k$ and in $H\cdot V_k$. Thus, by (3), the relative measure in J of

$$U_k+U_{k+1}+\cdots$$

is not greater than $2^{-1}\sum_{k=0}^{\infty} 1/n^k<1/2+1/n$, and the relative measure in J of

$$U_k-(G_{k+1}+G_{k+2}+\cdots)$$

is not less than $(1/2-1/n)(1-\sum_{k=1}^{\infty} 1/n^k)\geqq 1/2-3/n$. But, by (1)

$$J\cdot\{U_k-(G_{k+1}+G_{k+2}+\cdots)\}\subset J\cdot S\subset J\cdot\{U_k+U_{k+1}+\cdots\}.$$

So, the relative measure of S in J satisfies

$$(4)\qquad \frac{1}{2}-\frac{3}{n}\leqq\frac{m(S\cdot J)}{|J|}\leqq\frac{1}{2}+\frac{1}{n},$$

and since $n\to\infty$ as $|J|\to 0$, (4) shows that the metric density of S at x exists and is equal to $1/2$.

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NOTE ON ZEROS OF THE HERMITE POLYNOMIALS AND WEIGHTS FOR GAUSS' MECHANICAL QUADRATURE FORMULA

J. BARKLEY ROSSER

The main purpose of this note is to point out that, in most cases, use of the mechanical quadrature formula to compute integrals of the form

$$\int_{-\infty}^{\infty} e^{-x^2} f(x)\,dx \tag{1}$$

is unsatisfactory. One can easily see why this would be the case. In a *closed* interval one can approximate very closely to a continuous function by a high degree polynomial. However, outside the interval, the high degree polynomial will in most cases diverge very markedly from the given function. In (1), this divergence is eventually brought under control by the exponential, but often not in time to prevent a sizeable error.

We illustrate with a numerical example using coefficients from [1],[1] and from the end of section 22 of [2]. It can be shown that

$$\int_{-\infty}^{\infty} \frac{e^{-y^2}}{1+y^2}\,dy = 2\pi^{1/2}e\left\{\frac{\pi^{1/2}}{2} - \int_0^1 e^{-v^2}dy\right\}$$
$$= 1.34329\cdots$$

(see [2, Theorem 1–2 and formula (1–17)]). However the quadrature formulas for $n=2$, 10, and 16 give

$$n = 2, \quad 1.18164,$$
$$n = 10, \quad 1.34164,$$
$$n = 16, \quad 1.34313.$$

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

Thus, even for $n=16$, we have only a fair approximation, and the computation for $n=16$ is fairly laborious.

In [1], the quadrature formula was applied to

$$\int_{-\infty}^{\infty} e^{-x^2} \cos x dx$$

with considerable success. However, this is a very special case, since for this integral the asymptotic series actually converges, and rapidly enough that computation by the asymptotic series would be less laborious than using the quadrature formula. The asymptotic series for this integral is obtained by expanding $\cos x$ in the familiar Maclaurin series and integrating term by term.

It is probably true in most cases that if $f(x)$ is any decently differentiable function, then there will be less laborious means of computing (1) than numerical quadrature, so that numerical quadrature should be reserved for the case where $f(x)$ is known only through its values at some points. Further, one should not expect any great accuracy from the numerical quadrature in the majority of cases.

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NOTE ON A RESULT OF LEVINE AND LIFSCHITZ

I. I. HIRSCHMAN, JR. AND J. A. JENKINS

In a paper some years ago, [1],[1] Levine and Lifschitz considered, among other questions, the relationship between the gaps of a Fourier series and its admissible integral orders of zeros, a problem first treated by S. Mandelbrojt [2]. Let $f(t)$ be a function summable over the interval $(-\pi, \pi)$ and let

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt \tag{1}$$

be its associated Fourier series. Let $\psi(\alpha)$ be a continuous non-negative nondecreasing function defined for $0 \leqq \alpha \leqq 2\pi$. We shall say that $f(t)$ has a left-hand zero of integral order $\psi(\alpha)$ at $t=\pi$ if

$$\phi(\alpha) \equiv \int_{\pi-\alpha}^{\pi} |f(t)|\, dt \leqq \psi(\alpha) \qquad (0 \leqq \alpha \leqq 2\pi).$$

(One could equally well consider such a zero at any other point.) Natural choices for $\psi(\alpha)$ are:

a. $\psi(\alpha) = \alpha^n \qquad (n = 1, 2, \cdots),$

b. $\psi(\alpha) = e^{-(1/\alpha)^\rho} \qquad (0 < \rho < \infty),$

c. $\psi(\alpha) \begin{cases} = 0 & (0 \leqq \alpha \leqq \alpha_0), \\ > 0 & (\alpha_0 < \alpha \leqq 2\pi). \end{cases}$

If $\psi(\alpha) \not\equiv 0$, then there exists a number α_0 such that $\psi(\alpha) = 0$ $(0 \leqq \alpha \leqq \alpha_0)$ and $\psi(\alpha) > 0$ $(\alpha_0 < \alpha \leqq 2\pi)$. Let

$$r = -\log \psi(\alpha)/\alpha \qquad (\alpha_0 < \alpha \leqq 2\pi).$$

It is easily seen that for all r sufficiently large this equation may be inverted to give α as a function $\alpha = \eta(r)$ of r. It is $\eta(r)$ which we shall use as the measure of the zero of $f(t)$. It should be noted that such extreme behavior as a. and c. may be permitted and that our theorems are significant throughout this entire range.

Let $N(t)$ be the number of indices $n < t$ whose coefficients a_n and b_n are not both equal to zero. The density of these indices may be characterized by the function

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[1] Numbers in brackets refer to the references cited at the end of the paper

$$\zeta(r) = \log r + 2r^2 \int_0^\infty \frac{N(t)}{t(t^2 + r^2)}\, dt.$$

This slight correction to the original formula of Levine and Lifschitz was pointed out by S. Mandelbrojt in his review of their paper, see [3].

Levine and Lifschitz proved that if

$$\liminf_{r\to\infty} [3\zeta(2r) - r\eta(r)] = -\infty,$$

then $f(t) = 0$ almost everywhere in $(-\pi, \pi)$. In particular

$$\liminf_{r\to\infty} [c\zeta(r) - r\eta(r)] = -\infty$$

implies $f(x) = 0$ almost everywhere if $c = 12$, since $\zeta(\alpha r) \leqq \alpha^2 \zeta(r)$, $\alpha > 1$, r sufficiently large. In the other direction Levine and Lifschitz proved that this does not hold for $c < 1/2$. It is the object of this note to show that it does hold for $c = \pi$.

The method of proof is to construct the function

$$Q(z) = z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{n_k^2}\right)$$

where n_k are the values of the indices of the nonzero coefficients in the series (1). $Q(z)$ assumes its maximum modulus for a circle with center the origin on the imaginary axis and

$$\log |Q(iy)| = \zeta(y) \qquad (y > 0).$$

The integral function

$$F(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) e^{-izt}\, dt$$

has zeros at those integers where $Q(z)$ does not. Thus it is easily proved that the function

$$\Phi(z) = \frac{F(z)Q(z)}{\sin \pi z}$$

is integral and satisfies the bound

$$|\Phi(z)| < De^{\zeta(|z|)}$$

for all z, and for $y > 0$ the bound

$$|\Phi(z)| < De^{\zeta(|z|) - y\eta(y)} \qquad (z = x + iy = re^{i\theta}).$$

For these results we refer to [1].

Suppose that $\Phi(z) \not\equiv 0$; then $z=0$ is a zero of some finite order m, $0 \leqq m < \infty$. Let us regard the circle C: $|z| = r$. On this circle

$$\begin{aligned}\log \left| \Phi(z)/Dz^m \right| &\leqq \zeta(r) - m \log r \qquad (\pi \leqq \arg z \leqq 2\pi) \\ &\leqq \zeta(r) - m \log r - r \sin\theta\, \eta(r \sin\theta) \quad (0 \leqq \arg z \leqq \pi) \\ &\leqq \zeta(r) - m \log r - \sin\theta\, r\eta(r),\end{aligned}$$

since $\eta(r)$ is a nonincreasing function of r. Because the harmonic measure at the origin of an infinitesimal arc of length $rd\theta$ on the circle C is $(1/2\pi)d\theta$, we have, by an integrated form of the two constants theorem,

$$\begin{aligned}\log \left| \Phi(z)/Dz^m \right|_{z=1} &\leqq \frac{1}{2\pi}\int_0^{2\pi} [\zeta(r) - m \log r]d\theta - \frac{1}{2\pi}\int_0^{\pi} r\eta(r) \sin\theta d\theta \\ &\leqq \frac{1}{\pi}\left[\pi\zeta(r) - r\eta(r) - \pi m \log r\right].\end{aligned}$$

Because we have assumed

$$\liminf_{r\to\infty} \left[\pi\zeta(r) - r\eta(r)\right] = -\infty,$$

it follows that $[\Phi(z)/z^m]_{z=0} = 0$, a contradiction. Thus we must have $\Phi(z) \equiv 0$. Consequently $F(z) \equiv 0$ and $f(t) = 0$ almost everywhere in the interval $(-\pi, \pi)$.

$f(t)$ is said to have a two-sided zero of integral order $\psi(\alpha)$ at $t=\pi$ if

$$\phi(\alpha) = \int_{-\pi}^{-\pi+\alpha} |f(t)|\, dt + \int_{\pi-\alpha}^{\pi} |f(t)|\, dt \leqq \psi(\alpha) \qquad (0 \leqq \alpha \leqq \pi).$$

In this case a simple adaptation of the above argument shows that the condition

$$\liminf_{r\to\infty} \left[\frac{\pi}{2}\zeta(r) - r\eta(r)\right] = -\infty$$

is sufficient in order to have $f(t) = 0$ almost everywhere in $(-\pi, \pi)$.

If in the above conditions we replace lim inf by lim, more precise results may be obtained. Indeed if $f(t)$ has a two-sided zero at $t=\pi$, the condition

$$\lim_{r\to\infty} \left[(1+\epsilon)\zeta(r) - r\eta(r)\right] = -\infty \qquad (\epsilon > 0)$$

implies that $f(t) = 0$ almost everywhere in $(-\pi, \pi)$. This follows

from the fact that on the lines arg $z=\pi/2+\delta$, arg $z=-\pi/2-\delta$, we have the bound

$$|\Phi(z)| \leqq De^{\zeta(|z|)-|z|\cos\delta\,\eta(|z|\cos\delta)} \leqq De^{\zeta(|z|)-\cos\delta\,|z|\eta(|z|)}$$

so that if $\cos\delta \geqq (1+\epsilon)^{-1}$, $\Phi(z)$ tends to 0 as we go to infinity on these lines. Then, since $\Phi(z)$ is at most of order one, by the Phragmén-Lindelöf Principle it is bounded in the enclosed angle and this follows similarly for the angle $-\pi/2+\delta \leqq \arg z \leqq \pi/2-\delta$ and the two complementary angles. Thus $\Phi(z)\equiv 0$ and the result is proved.

If $f(t)$ has a one-sided zero at $t=\pi$, the same argument applied to $\Phi(z)\Phi(-z)$ shows that the condition

$$\lim_{r\to\infty} [(2+\epsilon)\zeta(r) - r\eta(r)] = -\infty \qquad (\epsilon > 0)$$

implies that $f(t)=0$ almost everywhere in $(-\pi, \pi)$.

It is easy to give examples which provide lower bounds for the constant c which may occur in the above result. Indeed, let $f_1(t)$ be identically zero from $-\pi+\delta$ to π and arbitrary (but not zero) in the rest of $(-\pi, \pi)$. We see at once that $\zeta(r)$ is asymptotically not greater than πr while for the choice $\psi(\alpha)=\phi(\alpha)$, $r\eta(r)$ is asymptotically equal to $(2\pi-\delta)r$. Thus the constant c cannot be taken less than 2. Similarly the function $f_2(t)$ identically zero outside $(-\delta, \delta)$, arbitrary in this interval, provides an example showing that for a two-sided zero the constant c cannot be taken as less than 1.

It should be mentioned that this does not, of course, supersede the result of Levine and Lifschitz since they showed that c cannot be taken less than 1/2 even when only very restricted classes of lacunary series are admitted.

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ON EULER TRANSFORMS

PHILIP HARTMAN AND AUREL WINTNER

Let E_0 denote the class of non-constant functions satisfying

$$d\phi(x) \geqq 0 \qquad \text{where } 0 \leqq x < \infty. \tag{1}$$

Then the Euler transform

$$\phi_\lambda(x) = \int_0^\infty (x+t)^{-\lambda} d\phi(t), \qquad \text{where } 0 < x < \infty, \tag{2}$$

is defined for some ϕ of class E_0. Let E_λ denote the class of all functions ϕ_λ belonging to a fixed $\lambda > 0$ and to some ϕ of class E_0. It will be shown that

$$E_\lambda \text{ is a (proper) subset of } E_\mu \text{ if } \lambda < \mu. \tag{3}$$

This implies that

$$E_\infty = \lim_{\lambda \to \infty} E_\lambda \text{ if } E_\infty = \sum_{0<\lambda<\infty} E_\lambda. \tag{4}$$

The class E_∞ is closely related to the Hausdorff-Bernstein class, consisting of all functions which are completely monotone for $0 < x < \infty$. Let E^∞ denote the latter class. It will be shown that

$$E_\infty \text{ is a (proper) subset of } E^\infty \tag{5}$$

and that, with reference to the "natural" topology on E^∞,

$$E_\infty \text{ is dense on } E^\infty. \tag{6}$$

It should be noted that E^∞ consists of all functions representable in the form

$$\phi^\infty(x) = \int_0^\infty e^{-xt} d\phi(t), \qquad \text{where } 0 < x < \infty, \tag{7}$$

provided that ϕ, instead of being subject to both restrictions (1), is subject only to the second of those restrictions and to the assumption that the integral (7) is convergent at every $x > 0$ (but not necessarily at $x = 0$). By the "natural" topology on E^∞ is meant that defined by the Helly convergence of monotone functions.

PROOF OF (3). It is readily verified from (1) and (2) that, as $x \to \infty$, no $\phi_\lambda(x)$ can tend to 0 as strongly as $x^{-\mu}$, if $\mu > \lambda$. On the

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other hand, (2) shows that $\phi_\mu(x)=x^{-\mu}$ if $\phi(t)=\operatorname{sgn} t$. Hence, the parenthetical assertion of (3) will need no further proof.

The main assertion of (3) is that, if $0<\lambda<\mu$, there belongs to every $\phi_\lambda(t)$ of class E_λ some $\phi^*(t)$ of class E_0 satisfying

$$\phi_\lambda(x) = \phi_\mu^*(x), \qquad \text{where } 0 < x < \infty. \tag{8}$$

It will be shown that such a $\phi^*(t)$ is supplied by the absolutely continuous function having the derivative

$$d\phi^*(t)/dt = A\int_0^t (t-s)^{\mu-\lambda-1}d\phi(s) \quad (0<t<\infty, \phi^*(0)=\phi^*(+0)), \tag{9}$$

where $A=A(\lambda, \mu)$ is a positive constant.

In view of (2), the assertion of (8) and (9) means that

$$\int_0^\infty (x+t)^{-\lambda}d\phi(t) = A\int_0^\infty (x+t)^{-\mu}\left\{\int_0^t (t-s)^{\mu-\lambda-1}d\phi(s)\right\}dt,$$

where $0<x<\infty$. It follows therefore from (1), and from (the Stieltjes form of) Fubini's theorem, that it is sufficient to verify the identity

$$\int_0^\infty (x+t)^{-\lambda}d\phi(t) = A\int_0^\infty \left\{\int_s^\infty (x+t)^{-\mu}(t-s)^{\mu-\lambda-1}dt\right\}d\phi(s).$$

But the latter holds for every ϕ of class E_0 if

$$(x+t)^{-\lambda} = A\int_t^\infty (x+s)^{-\mu}(s-t)^{\mu-\lambda-1}ds$$

is an identity in (x, t), where $x>0$, $t>0$. Hence, if the integration variable s is replaced by $s-t$, and if $x+t$ is then called x, it follows that it is sufficient to verify the identity

$$x^{-\lambda} = A\int_0^\infty (x+s)^{-\mu}s^{\mu-\lambda-1}ds,$$

where A is independent of x. Finally, the truth of this identity follows by changing s to xs (at a fixed $x>0$).

This proves (3). It also follows that the value of the constant $A=A(\lambda, \mu)$ which occurs in (9) is given by

$$1 = A\int_0^\infty (1+s)^{-\mu}s^{\mu-\lambda-1}ds.$$

Incidentally, the last integral can readily be transformed into the

integral defining $\mathrm{B}(\lambda, \mu) = \Gamma(\lambda)\Gamma(\mu-\lambda)/\Gamma(\mu)$; so that, in (9),

(9 bis) $$\mathrm{A} = \mathrm{A}(\lambda, \mu) = 1/\mathrm{B}(\lambda, \mu).$$

PROOF OF (5). It is seen from (1) that $(-1)^n$ times the nth derivative of the function (2) is non-negative for $n=0, 1, 2, \cdots$. This means that every class E_λ is contained of the Hausdorff-Bernstein class, E^∞. Hence, by (4), E_∞ is contained in E^∞. The parenthetical part of (5) follows from the fact that e^{-x} is in E^∞, but e^{-x} is not in E_∞. For otherwise e^{-x} would be in E_λ some λ, which would imply, for $x>0$ and $t_0>0$, that

$$e^{-x} \geqq (x + t_0)^{-\lambda}(\phi(t_0) - \phi(0)),$$

by (1) and (2). If t_0 is chosen so that $\phi(t_0) - \phi(0) > 0$, the last formula line leads to a contradiction for large x. This contradiction shows that e^{-x} cannot be in E_∞ and completes the proof of (5).

PROOF OF (6). Let $b>0$ and $\lambda>0$. Then it is readily verified that

$$b^\lambda \int_0^\infty (x + t)^{-\lambda} d \operatorname{sgn} (t - b) = (1 + x/b)^{-\lambda},$$

where $0 \leqq x < \infty$. Clearly, the expression on the left of this identity represents a function of class E_λ. On the other hand, the expression on the right tends to e^{-ax} if $\lambda \to \infty$ and $b = \lambda/a$, where a is any positive constant. Accordingly, every function of the form e^{-ax} is a limit, as $\lambda \to \infty$, of functions contained in E_λ. Hence, the same is true of every function of the form

$$\sum_{k=1}^{m} c_k e^{-a_k x},$$

where c_k, a_k are arbitrary non-negative constants. Since the latter sum is identical with the case

$$\phi(x) = \sum_{a_k \leqq x} c_k$$

of the transform (7), the assertion of (6) now follows by a standard application of Helly's theorems on monotone functions.

THE JOHNS HOPKINS UNIVERSITY

CONVERGENCE FIELDS OF ROW-FINITE AND ROW-INFINITE TOEPLITZ TRANSFORMATIONS

P. ERDŐS AND G. PIRANIAN[1]

During a recent conversation, R. P. Agnew suggested a determination of the validity of the proposition that row-infinite Toeplitz transformations are more powerful than row-finite transformations. Before this proposition is examined, it is necessary to assign a precise meaning to it. Corresponding to every sequence s a regular row-finite Toeplitz transformation A can be constructed such that the transform As converges to a finite limit. In order to be of interest, the proposition must therefore be interpreted in terms of convergence fields of individual transformations. Also, because every row-infinite regular Toeplitz matrix is the sum of two Toeplitz matrices A and B, where A is regular and row-finite and B has the norm zero, the convergence field in the space of bounded sequences of every regular Toeplitz transformation coincides with that of a row-finite regular transformation; in other words, the problem is trivial except in its reference to unbounded sequences.

THEOREM 1. *There exists a regular Toeplitz transformation whose convergence field is not contained in the convergence field of any regular row-finite Toeplitz matrix.*

Let A be the matrix

$$\begin{Vmatrix} 1/2 & 0 & 1/4 & 0 & 1/8 & 0 & 1/16 & \cdots \\ 0 & 1/2 & 0 & 0 & 0 & 1/4 & 0 & \cdots \\ 0 & 0 & 0 & 1/2 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{Vmatrix}$$

whose elements a_{nk} $(n=1, 2, \cdots; k=1, 2, \cdots)$ are given by the rule

$$a_{nk} = 1/2^p \qquad (n = 1, 2, \cdots; k = 2^{n-1}(2p-1); p = 1, 2, \cdots),$$

$$a_{nk} = 0 \qquad \text{otherwise.}$$

Corresponding to every regular row-finite matrix B a sequence s will be constructed such that the sequence As converges to zero while the sequence Bs diverges. The construction hinges on the fact, readily

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shown, that each column of A contains precisely one positive element; and it will be carried out in two ways, according as the matrix B falls into one or the other of two classes.

In Case I there exists, for every positive integer m, a nonzero element b_{nk}, well to the right in its row, such that the positive element in the kth column of A occurs below the first m rows. To be precise: to each pair of positive integers m and h corresponds a pair of integers n and k $(k>h)$ such that

(i) $$b_{nk} \neq 0,$$

(ii) $$a_{rk} = 0 \qquad (r = 1, 2, \cdots, m).$$

Let $b(n_1, k_1)$ be a nonzero element in B; let

$$s_k = 0 \qquad (k = 1, 2, \cdots, k_1 - 1),$$

$$s_k = 1/b(n_1, k_1) \qquad (k = k_1),$$

$$s_k = 0 \qquad (k = k_1 + 1, k_1 + 2, \cdots, k_1'),$$

where the index k_1' is large enough so that $b(n_1, k) = 0$ whenever $k > k_1'$. It is then certain that the n_1st element of the sequence Bs has the value one. On the other hand, there exist integers k_1'' $(k_1'' > k_1')$ and n_1' such that $a(n_1', k_1'') \neq 0$. If

$$s_k = 0 \qquad (k = k_1' + 1, k_1' + 2, \cdots, k_1'' - 1),$$

s_k has an appropriate value for $k = k_1''$,

$$s_k = 0 \qquad \text{for all } k\,(k > k_1'') \text{ for which } a(n_1', k) > 0,$$

then all except finitely many of the partial sums of the series $\sum a(n_1', k)s_k$ are zero.

Suppose that the element s_k of the sequence s has been defined for $k = 1, 2, \cdots, k_{r-1}''$. Let $b(n_r, k_r)$ be a nonzero element in B, subject to the consistent conditions on the indices that $k_r > k_{r-1}''$, $n_r > n_{r-1}$, and $a(n, k_r) = 0$ when $n \leqq n_{r-1}$; and let $a(n_r', k_r)$ be the positive element in the k_rth column of A. Let k_r' be an integer large enough so that $b(n_r, k) = 0$ when $k > k_r'$, and k_r'' an integer greater than k_r' and such that the element $a(n_r', k_r'')$ is positive. The elements s_k are chosen to be zero for $k_{r-1}'' < k < k_r$ and for $k_r < k < k_r''$; the elements s_k $(k = k_r$ and $k = k_r'')$ are chosen so that the n_rth element of the sequence Bs has the value $(-1)^{r+1}$ and the k_r''th partial sum of the series $\sum a(n_r', k)s_k$ is zero; and the elements s_k $(k > k_r'')$ for which $a(n_r', k) > 0$ are chosen to be zero.

Let this construction of the sequence s be continued indefinitely. Because each column of A contains only one positive element, the multiple definition that is accorded to some of the elements s_k is only

apparent: all definitions assign the value zero to such elements. The transform As is the sequence 0, 0, $\cdots$, while the transform Bs contains infinitely many elements of value 1 and -1, respectively.

Case II is that in which there exists a pair of integers N and K such that b_{nk} is zero whenever the three conditions $a_{nk}>0$, $n>N$, $k>K$ are satisfied. In this case, the sequence s to be constructed is subjected to the condition that $s_k=0$ whenever $k\leqq K$ or one of the elements a_{nk} $(n>N)$ is positive. This condition implies that all partial sums of the series $\sum a_{nk}s_k$ $(n=N+1, N+2, \cdots)$ are zero. The task at hand will be accomplished when the remaining elements s_k are chosen so that the transform Bs diverges and each of the N series $\sum a_{nk}s_k$ $(n=1, 2, \cdots, N)$ converges to some finite value a_n.

Let (n_1, k_1), (n_2, k_2), $\cdots$ be a sequence of pairs of integers such that

(i) $$n_1 > N,$$

(ii) $$b(n_r, k) = 0 \qquad (k = k_r + 1, k_r + 2, \cdots),$$

(iii) $$\sum_{k=1}^{k_{r-1}} |b(n_r, k)| < 1/2^r \qquad (r = 2, 3, \cdots).$$

The elements s_k of the sequence s which have not yet been chosen are now defined to have the value $(-1)^r$ when the index k is subject to the inequality $k_{r-1}<k\leqq k_r$ $(r=2, 3, \cdots)$. Then, with the notation $Bs=t$, the equation $\lim [t(n_r)-(-1)^r]=0$ is satisfied. On the other hand, the N series $\sum a_{nk}s_k$ $(n=1, 2, \cdots, N)$ converge absolutely, and the proof of Theorem 1 is complete.

The proof that has just been given needs only minor modifications to lead to the following result.

THEOREM 2. *If A is a regular Toeplitz matrix with the property that each row of A contains infinitely many nonzero elements and each column of A contains precisely one nonzero element, the convergence field of A is not contained in that of any row-finite matrix.*

Theorems 1 and 2 assert the existence of row-infinite Toeplitz matrices that are in some sense more powerful than any row-finite matrices. But this superiority is not a uniform property of row-infinite matrices.

THEOREM 3. *There exists a regular Toeplitz matrix whose elements are all positive and whose convergence field is identical with that of the Hölder transformation.*

Let A be a the matrix

$$\left\|\begin{array}{cccccc} 1 & 1/2! & 1/3! & 1/4! & \cdots \\ 1 & 1/2^2 & 1/2^3 & 1/2^4 & \cdots \\ 1/2 & 1/2 & 1/3! & 1/4! & \cdots \\ 1/2 & 1/2 & 1/2^3 & 1/2^4 & \cdots \\ 1/3 & 1/3 & 1/3 & 1/4! & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right\|.$$

As is seen by examining separately the elements below and above the broken line, this matrix is the sum of the Hölder matrix (with each row in duplicate) and another matrix whose elements b_{nk} are zero when $2k \leqq n$ and whose remaining elements tend to zero uniformly with respect to n as k becomes large; the rapidity with which the elements b_{nk} tend to zero depends only on the parity of n. If the sequence s is evaluated by the Hölder transformation, $s_n = o(n)$, and therefore the series

$$\sum_{k=1}^{\infty} \frac{s_k}{k!} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{s_k}{2^k}$$

converge. This implies that the sums

$$\sum_{k=r}^{\infty} \frac{s_k}{k!} \quad \text{and} \quad \sum_{k=r}^{\infty} \frac{s_k}{2^k}$$

tend to zero as the lower limit r becomes large, and it follows that the sequence As converges to the same limit as the sequence Hs.

On the other hand, suppose that the sequence s is not evaluated by the Hölder transformation, but is evaluated to zero by the matrix A. Then

$$\limsup_{r \to \infty} \left| \sum_{k=r}^{\infty} \frac{s_k}{k!} \right| > 0,$$

and therefore

$$\limsup_{k \to \infty} |s_k| / (k-2)! > 0.$$

The last inequality implies that

$$\limsup_{k \to \infty} |s_k| / 2^k > 0,$$

that is, that the series $\sum a_{nk} s_k$ does not converge when n is even. In

other words, the transform As cannot exist, contrary to the supposition that it converges.

THEOREM 4. *There exists a regular row-finite Toeplitz matrix whose convergence field is not contained in the convergence field of any row-infinite matrix.*

Let A be the matrix

$$(1) \qquad \left\| \begin{matrix} 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/2 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1/2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix} \right\|.$$

That its existence proves Theorem 4 is a consequence of the following result.

THEOREM 5. *For the convergence field of a regular Toeplitz matrix B to contain the convergence field of the matrix* (1) *it is necessary and sufficient that B be row-finite and that there exist a constant k_0 such that the relation $b_{n,2k+1}=b_{n,2k+2}$ holds for all n whenever $k>k_0$.*

If B is not row-finite, there exists a sequence s whose transform by A is the sequence $0, 0, 0, \cdots$ and whose transform by B does not exist. If B is row-finite and there exists an infinite increasing sequence of integers k_r such that for some integer n $(n=n_r)$ the relation

$$b(n_r, 2k_r + 1) - b(n_r, 2k_r + 2) \neq 0$$

holds, it is possible to construct a sequence s which satisfies the conditions

$$s_{2k+1} = -s_{2k+2} \qquad (k = 1, 2, \cdots),$$
$$t(n_r) = (-1)^r \qquad (t = Bs).$$

Because B is row-finite, the sequence $\{n_r\}$ tends to infinity, and the necessity of the conditions follows.

Suppose, on the other hand, that B satisfies the condition of the theorem and that A evaluates the sequence s to zero. Then

$$s_{2k+1} = -s_{2k+2} + \epsilon_k$$

where ϵ is a null sequence. That B evaluates s to zero follows from the fact that the Toeplitz transformations are linear.

SYRACUSE UNIVERSITY AND
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LINEAR FUNCTIONALS ON CERTAIN BANACH SPACES

E. J. MCSHANE

1. Introduction. The purpose of this note is to present what is believed to be a simple proof of a theorem on the representation of linear functionals on spaces L_p $(p>1)$ and some generalizations of such spaces. Instead of making use of the Radon-Nikodym theorem, the proof utilizes some simple consequences of uniform convexity, and applies at once to spaces L_p over arbitrary sets X on a family of whose subsets a measure function is defined, no decomposition of the space being involved. The proofs are stated for complex Banach spaces; the real case allows some obvious simplifications.

2. Linear functionals and derivatives of norms.

LEMMA 1. *Let g be an element of a complex Banach space B and L a linear continuous functional such that $L(g)=|L|\cdot\|g\|$. Then for each f in B we have*

$$|L|\,D^-(\|g+tf\|)_{t=0} \leqq \mathcal{R}L(f) \leqq |L|\,D_+(\|g+tf\|)_{t=0}. \tag{1}$$

If $|L|$ or $\|g\|$ is 0, this is trivial. If neither of these is 0, we find that there is no loss of generality in assuming that $|L|=\|g\|=1$; in fact, we shall apply the lemma only to cases in which $\|g\|=1$. For all complex z and all f in B we have $L(g+z[f-L(f)g])=L(g)=1=|L|$, so

$$\|g+z[f-L(f)g]\| \geqq 1 \qquad \text{for all complex } z. \tag{2}$$

The equation

$$g+tf=(1+tL(f))\{g+t[1+tL(f)]^{-1}(f-L(f)g)\}$$

is an identity for $t\neq -1/L(f)$, and by (2) the norm of the vector in braces is never less than 1, which is its value at $t=0$. Hence for all real t near 0 we have

$$\|g+tf\|-\|g\| \geqq |1+tL(f)|-1 \geqq t(\mathcal{R}L(f)).$$

Dividing by negative t and letting t approach zero yields the first of inequalities (1); with positive t we obtain the other inequality.

LEMMA 2. *Let g be a nonzero element of a complex Banach space B, and let L be a linear continuous functional on B such that $L(g)=|L|\cdot\|g\|$. Assume that for each f in B the function defined for all real t by*

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the expression $\|g+tf\|$ *is differentiable at* $t=0$. *Then for all* f *the equation*

$$L(f) = |L| \frac{d}{dt} [\|g + tf\| + i\|g - itf\|]_{t=0} \tag{3}$$

is satisfied.

Under the present hypotheses, the conclusion of Lemma 1 becomes an equation. Writing this for f and for $-if$, we obtain

$$\Re L(f) = |L| \frac{d}{dt} \|g + tf\|_{t=0},$$

$$\Im L(f) = \Re L(-if) = |L| \frac{d}{dt} \|g - itf\|_{t=0},$$

establishing (3).

Next we recall [1, p. 396][1] that the space B is *uniformly convex* if to each positive ϵ there corresponds a positive $\delta(\epsilon)$ such that if $\|f\| = \|g\| = 1$ and $\|f-g\| \geqq \epsilon$, then

$$\|(f + g)/2\| \leqq 1 - \delta(\epsilon). \tag{4}$$

LEMMA 3. *If* B *is uniformly convex, to each linear continuous functional* L *on* B *there corresponds a unit vector* g_L *such that*

$$|L| D^-\|g_L + tf\| \leqq \Re L(f) \leqq |L| D_+\|g_L + tf\|$$

for all f *in* B. *In particular, if* $\|g_L+tf\|$ *is a differentiable function of* t *at* $t=0$ *for all* f *in* B, *the equation*

$$L(f) = |L| \frac{d}{dt} (\|g_L + tf\| + i\|g_L - itf\|)_{t=0} \tag{5}$$

holds for all f *in* B.

Choose a sequence of unit vectors $f_1, f_2, \cdots$ such that $L(f_n)$ tends to $|L|$. Let ϵ be positive; then so is $\delta(\epsilon)$, and for all large m and n we have

$$L([f_m + f_n]/2) = [L(f_m) + L(f_n)]/2 > |L| [1 - \delta(\epsilon)],$$

so that $\|(f_m+f_n)/2\| > 1-\delta(\epsilon)$. By the definition of uniform convexity, this implies that $\|f_m-f_n\| < \epsilon$, and the sequence converges. Let g_L be its limit; this is a unit vector. Also

$$L(g_L) = |L| = |L| \cdot \|g_L\|,$$

[1] Numbers in brackets refer to the bibliography at the end of the paper.

so the hypotheses of Lemma 1 are satisfied, and the conclusion follows from Lemmas 1 and 2.

3. **Spaces $L_p(B_1)$.** Now let m be a non-negative measure function on a σ-field of subsets of a space X. A function f on X to a Banach space B_1 is said to belong to the space $L_p(B_1)$ if $\|f(\cdot)\|_1^{p-1}f(\cdot)$ is summable in the sense of Bochner over X, where $\|\cdot\|_1$ denotes the norm in the space B_1. (If the reader is interested only in the usual space L_p, he should interpret B_1 as the real or complex number system, $\|\cdot\|_1$ as absolute value, and the integral as the Lebesgue integral.) Assume $p>1$ and define as usual $p'=p/(p-1)$. It is then known that if B_1 is uniformly convex (in particular, if it is the real or complex number system), $L_p(B_1)$ is also uniformly convex [2, p. 504]; we append another proof of this fact to the present note.

THEOREM I. *Let B_1 be a uniformly convex Banach space such that for each nonzero g in B_1 and each f in B_1, the function $=\|g+tf\|_1$ is differentiable at $t=0$. Let p be greater than 1, and let L be a functional linear and continuous on $L_p(B_1)$. Then there exists a unit vector g_L in $L_p(B_1)$ such that for every f in $L_p(B_1)$ we have*

$$L(f) = |L| \int_X \|g_L(x)\|_1^{p-1} \frac{d}{dt} [\|g_L(x) + tf(x)\|_1 + i\|g_L(x) - itf(x)\|_1]_{t=0} m(dx), \tag{6}$$

wherein the derivative in the integrand is to be assigned any arbitrary finite value when the factor multiplying it has the value 0.

Let g_L be the unit vector in $L_p(B_1)$ whose existence is guaranteed by Lemma 3, and let f belong to $L_p(B_1)$. The equation

$$\frac{d}{dt}\|g_L(x) + tf(x)\|_1^p = p\|g_L(x) + tf(x)\|_1^{p-1} \frac{d}{dt}\|g_L(x) + tf(x)\|_1$$

holds whenever $g_L(x)+tf(x)\neq 0$ by elementary differentiation, and when that vector is 0 it continues to hold if the derivative in the right member is assigned any finite value. For $|t|<1$ the right member cannot exceed the function $p[\|g_L(x)\|_1+\|f(x)\|_1]^p$, which is summable. This permits us to differentiate under the integral sign in the integral which defines $\|g_L+tf\|$; recalling that $\|g_L\|=1$, we obtain

$$\frac{d}{dt}\|g_L + tf\|_{t=0} = \int_X \|g_L(x)\|_1^{p-1} \left[\frac{d}{dt}\|g_L(x) + tf(x)\|_1\right]_{t=0} m(dx). \tag{7}$$

Thus the hypotheses of Lemma 2 are satisfied, and the conclusion of

Lemma 2 is equation (6).

As a special case, we obtain the representation of the general linear continuous functional on complex L_p.

THEOREM II. *Let L be a continuous linear functional on the complex Banach space L_p of functions on X, wherein $p>1$. Then there exists an element g of $L_{p'}$, where $p'=p/(p-1)$, such that the norm of g in $L_{p'}$ is $|L|$ and*

$$L(f) = \int_X g(x)f(x)m(dx) \tag{8}$$

for all f in L_p.

If a and b are complex numbers such that $a\neq 0$, from

$$\begin{aligned} |a+tb| &= |a|\,[|1+t(b/a)|\,] \\ &= |a|\,[(1+\Re tb/a)^2+(\Im tb/a)^2]^{1/2} \end{aligned}$$

we obtain

$$[d|a+tb|/dt]_{t=0} = |a|\Re(b/a) = \Re\bar{a}b/|a|,$$

whence

$$\begin{aligned} &[d|a+tb|/dt + id|a-itb|/dt]_{t=0} \\ &\qquad = \Re[\bar{a}b/|a|\,] + i\Im[\bar{a}b/|a|\,] = \bar{a}b/|a|. \end{aligned} \tag{9}$$

Thus the integrand in (6) takes the form

$$\|g_L(x)\|^{p-2}\overline{g_L(x)}f(x)$$

whenever $g_L(x)\neq 0$. If we now define

$$\begin{aligned} g(x) &= |L|\cdot\|g_L(x)\|^{p-2}\overline{g_L(x)} && \text{where} \quad g_L(x)\neq 0, \\ g(x) &= 0 && \text{wherever } g_L(x)=0, \end{aligned}$$

we see that (8) holds. Now $g(x)$ is everywhere a non-negative multiple of the conjugate of $g_L(x)$, and by the definition of p' we have

$$|g(x)|^{p'} = |L|^{p'}|g_L(x)|^p.$$

Hence g is in $L_{p'}$, and its norm in $L_{p'}$ is $|L|$, completing the proof.

THEOREM III. *Let μ be a measure function on a σ-field of subsets of a set Y, and let m be a measure function on a σ-field of subsets of a set X. Let L_q be the space of complex functions on Y such that $|f(y)|^{q-1}f(y)$ is summable over Y, and $L_p(L_q)$ the space of functions on X to L_q such that $\|f(x)\|_q^{p-1}f(x)$ is Bochner summable over X, the symbol $\|\cdot\|_q$ denoting*

the norm in L_q. Assume p and q greater than 1, and define $p'=p/(p-1)$, $q'=q/(q-1)$. Let L be a linear continuous functional on the complex Banach space $L_p(L_q)$. Then there exists an element g of $L_{p'}(L_{q'})$, with norm $|L|$ in that space, such that for every element

$$f=(f(x)\mid x\in X)=((f(y,x)\mid y\in Y)\mid x\in X)$$

of the space $L_p(L_q)$ we have

$$L(f)=\int_X\left\{\int_Y g(y,x)f(y,x)\mu(dy)\right\}m(dx). \tag{10}$$

For L we first use equation (6), recalling that for each x the value of $g_L(x)$ is the function $(g_L(y,x)\mid y\in Y)$. For the derivative in the integrand of (6) we substitute from (7), applied to the space Y instead of X. The inner integrand will contain a derivative, which we replace by its value as computed in (9). The result is

$$L(f)=|L|\int_X\left\{\|g_L(x)\|^{p-1}\int_Y|g_L(y,x)|^{q-2}\overline{g_L(y,x)}f(y,x)\mu dy\right\}m(dx),$$

where the inner integrand is to be understood to mean 0 whenever $g_L(y,x)$ vanishes. We define

$$g(y,x)=|L|\,\|g_L(x)\|^{p-1}\,|g_L(y,x)|^{q-2}\overline{g_L(y,x)}$$

wherever $g_L(y,x)\neq 0$, and set $g(y,x)=0$ where $g_L(y,x)=0$. Then (10) holds. The verification of the other statements about g is much the same as in the proof of Theorem II.

In Theorem II we extended the representation theorem for linear functionals on L_p to maximum generality as regards the space X. It is not possible to extend the theorem on the conjugate space of L_1 to a corresponding generality, as is shown by the following example, which is the result of a conversation with T. A. Botts and V. L. Klee. Let X consist of the points of the Euclidean plane. Let the exterior measure of a countable set E be 1 or 0 according as the origin is or is not in E; the exterior measure of every uncountable set is $+\infty$. This exterior measure generates a Carathéodory measure which coincides with itself, all sets being measurable. A function f is summable if and only if it is zero except on a countable set and $f(0)$ is finite, and in this case its integral is $f(0)$. All spaces L_p $(p\geqq 1)$ coincide, and the norm is $\|f(\cdot)\|=|f(0)|$, and all are conjugate to each other. But the space conjugate to L_1 is L_1, which is distinct from the space of essentially measurable functions on X.

4. **Proof of uniform convexity of spaces $L_p(B)$.** We now prove that

when $L_p(B)$ is defined as above, and B is uniformly convex and $p>1$, the space $L_p(B)$ is also uniformly convex. For the norms in B and $L_p(B)$ we shall use the symbols $\|\cdot\|$, $\|\cdot\|_p$ respectively. As a first step, we show that to each positive ϵ corresponds a positive $\delta(\epsilon)$ such that if f and g are in B, and $1=\|f\|\geqq\|g$, and $\|f-g\|\geqq\epsilon$, then

$$\|(f+g)/2\|^p \leqq [1-\delta(\epsilon)][\|f\|^p+\|g\|^p]/2. \tag{11}$$

Assume this false; then there exist $\epsilon>0$ and sequences f_n, g_n of elements of B such that $1=\|f_n\|\geqq\|g_n\|$, $\|f_n-g_n\|\geqq\epsilon$, and

$$\lim_{n\to\infty}\frac{\|(f_n+g_n)/2\|^p}{[\|f_n\|^p+\|g_n\|^p]/2}=1. \tag{12}$$

From the convexity of t^p $(0\leqq t\leqq 1)$ we obtain

$$([1+t]/2)^p<(1+t^p)/2 \qquad \text{for } 0\leqq t<1 \tag{13}$$

so by (12) and the triangle inequality

$$\lim_{n\to\infty}\frac{(1+\|g_n\|)^p 2^{p-1}}{1+\|g_n\|^p}=1.$$

But by (13) this implies $\lim\|g_n\|=1$. Define $u_n=g_n/\|g_n\|$; then $\lim\|u_n-g_n\|=0$, so $\liminf\|u_n-f_n\|=\liminf\|g_n-f_n\|\geqq\epsilon$, and by (12)

$$\lim\|(f_n+u_n)/2\|=1.$$

This contradicts the uniform convexity of B.

As a corollary, if f and g are not both 0 we have

$$\|(f+g)/2\|^p \leqq [1-\delta(\|f-g\|/\sup(\|f\|,\|g\|))][\|f\|^p+\|g\|^p]/2. \tag{14}$$

Now assume that $f(\cdot)$ and $g(\cdot)$ are in $L_p(B)$, and that ϵ is positive, and that $\|f\|=\|g\|=1$ and $\|f-g\|_p\geqq\epsilon$. Let E be the subset of X on which

$$\begin{aligned}\|f(x)-g(x)\|^p &\geqq (\epsilon^p/4)(\|f(x)\|^p+\|g(x)\|^p)\\ &\geqq (\epsilon^p/4)\sup(\|f(x)\|^p,\|g^p(x)\|).\end{aligned} \tag{15}$$

On this set we have, by (14)

$$\|[f(x)+g(x)]/2\|^p \leqq [1-\delta(\epsilon/4^{1/p})][\|f(x)\|^p+\|g(x)\|^p]/2. \tag{16}$$

The integral of $\|f-g\|^p$ over $X-E$ is at most $\epsilon^p/2$, so

$$\int_E\|f(x)-g(x)\|^p m(dx)\geqq\epsilon^p/2. \tag{17}$$

Thus the functions which coincide with $f(\cdot)$ and $g(\cdot)$ on E and vanish on $X-E$ have distance at least $\epsilon/2^{1/p}$. So at least one of them has distance not less than $\epsilon/2\cdot 2^{1/p}$ from the origin, and

$$\sup\left(\int_E \|f(x)\|^p m(dx),\ \int_E \|g(x)\|^p m(dx)\right) \geqq \epsilon^p/2^{p+1}. \tag{18}$$

Now by (16) and (18),

$$\int_X [(\|f(x)\|^p + \|g(x)\|^p)/2]m(dx) - \int_X [\|(f(x)+g(x)\|/2]^p m(dx)$$

$$\geqq \int_E [(\|f(x)\|^p + \|g(x)\|^p)/2]m(dx) - \int_E [\|f(x)+g(x)\|/2]^p m(dx)$$

$$\geqq \int_E \delta(\epsilon/4^{1/p})\{[\|f(x)\|^p + \|g(x)\|^p]/2\}m(dx)$$

$$\geqq \delta(\epsilon/4^{1/p})(\epsilon^p/2^{p+2}),$$

whence

$$\|f(\cdot)+g(\cdot)\|_p \leqq [1-\delta(\epsilon/4^{1/p})(\epsilon^p/2^{p+2})]^{1/p}. \tag{19}$$

If we define $\delta_1(\epsilon)$ to be the difference between 1 and the right member of (19), we see that $\delta_1(\epsilon)>0$, and $L_p(B)$ is uniformly convex.

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NOTE ON A THEOREM OF KAKUTANI

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1. Introduction. Limit theorems of probability have attracted much attention for a number of years. For a wide class of limit theorems, the language and methods of infinite product measure spaces have proved to be appropriate. It is the purpose of this note to discuss the application of a recent theorem of Kakutani on the equivalence of infinite product measures to the problem of extending known sufficient conditions for such theorems.

The distribution function of each of a sequence of independent real-valued random variables generates a measure on R_1, the space of real numbers. These measures determine a direct infinite product measure on the product space $R_1 \times R_1 \times \cdots$. Kakutani's theorem gives necessary and sufficient conditions for the equivalence of two direct infinite product measures. If the infinite product measures associated with two sequences of random variables are equivalent, then they both obey, or both fail to obey, any of a wide class of limit theorems. Thus if one sequence of independent random variables satisfies known sufficient conditions for such a limit theorem, and the two associated product measures satisfy Kakutani's conditions for equivalence, then the other sequence also obeys the limit theorem. In this way Kakutani's theorem provides a device for extending known sufficient conditions. In its present form, it does not contain a certain well known device of this type. Indeed, for this purpose, full equivalence of the product measures is not in general necessary. However, it is possible to derive, from Kakutani's conditions, conditions apparently less restrictive, which will permit the inclusion of this device as a special case.

2. Kakutani's theorem. For each positive integer n, let $\mathcal{B}_n$ be a (countably) additive class of subsets of a set Ω_n, and let m_n be a (countably additive) measure on $\mathcal{B}_n$, such that $m_n(\Omega_n) = 1$. Let Ω^* be the infinite product space $\Omega_1 \times \Omega_2 \times \cdots$ of all sequences $\omega^* = \{\omega_n\}$, $\omega_n \in \Omega_n$ $(n = 1, 2, \cdots)$. Let $\mathcal{R}^*$ be the family of all rectangular sets R^*,

$$R^* = B_1 \times B_2 \times \cdots \times B_k \times \Omega_{k+1} \times \Omega_{k+2} \times \cdots,$$

where $B_i \in \mathcal{B}_i$ $(i = 1, 2, \cdots, k; 1, 2, \cdots)$. Let $\mathcal{B}^*$ be the additive class of sets of Ω^* generated by $\mathcal{R}^*$, and m^* the measure [1, 2; cf.

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also 3][1] on $\mathcal{B}^*$ such that $m^*(R^*) = \prod_{n=1}^{k} m_n(B_n)$.

The condition $m_n(\Omega_n) = 1$ may be replaced by $0 < \prod_1^k m_n(\Omega_n) < \infty$, a measure m^* then being determined on $\mathcal{B}^*$ such that $m^*(R^*) = \prod_1^k m_n(B_n) \prod_{k+1}^{\infty} m_n(\Omega_n)$. It is sufficient to consider the measures $\mu_n \equiv m_n/m_n(\Omega_n)$, which satisfy $\mu_n(\Omega_n) = 1$, and to define $m^*(B^*) = \mu^*(B^*) \prod_1^{\infty} m_n(\Omega_n)$ for $B^* \in \mathcal{B}^*$.

One may define similarly $\Omega_k^* = \Omega_k \times \Omega_{k+1} \times \cdots$, $\mathcal{R}_k^*$, $\mathcal{B}_k^*$, and m_k^* $(k = 1, 2, \cdots)$.

LEMMA 1. *If*

$$E^* = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{k-1} \times E_k^*, \qquad E_k^* \in \mathcal{B}_k^*,$$

then $E^* \in \mathcal{B}^*$, *and* $m^*(E^*) = m_1(\Omega_1) m_2(\Omega_2) \cdots m_{k-1}(\Omega_{k-1}) m_k^*(E_k^*)$ [4, p. 145].

DEFINITION. Let $\mathcal{B}$ be an additive family of subsets of set Ω of elements ω. Measures m, m' defined on $\mathcal{B}$ are *equivalent* $(m \sim m')$ on $\mathcal{B}$ if each is absolutely continuous with respect to the other; that is, if $m(B) = 0$ implies $m'(B) = 0$ for $B \in \mathcal{B}$, and conversely. It follows from a theorem of Radon-Nikodym (cf. S. Saks, *Theory of the integral*, 1937, p. 36) that there exists a $\mathcal{B}$-measurable function $\phi(\omega)$ defined on Ω such that

$$m'(B) = \int_B \phi(\omega) dm(\omega) \qquad \text{for } B \in \mathcal{B},$$

and a function $\psi(\omega)$ such that $m(B) = \int_B \psi(\omega) dm'(\omega)$, for $B \in \mathcal{B}$.

Let m_n, m_n' be two additive measures on $\mathcal{B}_n$ $(n = 1, 2, \cdots)$.

THEOREM 1. (KAKUTANI'S THEOREM [1]). *If*

$$m_n \sim m_n' \qquad \text{on } \mathcal{B}_n, \tag{1}$$

$$m_n(\Omega_n) = m_n'(\Omega_n) = 1, \tag{2}$$

then a necessary and sufficient condition that $m^* \sim m^{*\prime}$ *on* $\mathcal{B}^*$ *is that*

$$\prod_{n=1}^{\infty} \rho_n > 0, \tag{3}$$

where

$$\rho_n = \rho(m_n, m_n') = \int_\Omega (\phi_n(\omega))^{1/2} dm_n(\omega), \tag{4}$$

and where $\phi_n(\omega)$ *is such that*

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$(5) \qquad m_n'(B) = \int_B \phi_n(\omega) dm_n(\omega) \qquad \text{for } B \in \mathfrak{B}_n \ (n = 1, 2, \cdots).$$

An immediate consequence of Kakutani's theorem is the following lemma.

LEMMA 2. *If* (1) *holds* $(n=1, 2, \cdots)$,

$$(6) \qquad m_n(\Omega_n) \leqq 1, \quad m_n'(\Omega_n) \leqq 1 \qquad (n = 1, 2, \cdots),$$

$$(7) \qquad \prod_1^\infty m_n(\Omega_n) > 0, \quad \prod_1^\infty m_n'(\Omega_n) > 0,$$

and (3) *holds, then* $m^* \sim m^{*\prime}$ *on* $\mathfrak{B}^*$.

It is sufficient to apply Kakutani's theorem to the measures $\mu_n(B) = m_n(B)/m_n(\Omega_n)$ and $\mu_n'(B) = m_n'(B)/m_n'(\Omega_n)$ $(B \in \mathfrak{B}_n)$. For, then

$$\rho(\mu_n, \mu_n') = \rho(m_n, m_n')/(m_n(\Omega_n) m_n'(\Omega_n))^{1/2},$$

hence $\prod_1^\infty \rho(\mu_n, \mu_n') > 0$ (note that by Schwarz' inequality $\rho(\mu_n, \mu_n') \leqq 1$). Thus $\mu^* \sim \mu^{*\prime}$, and $m^* \sim m^{*\prime}$ (p. 410).

Let $\mathfrak{F}^*$ denote the family of all sets of $\mathfrak{B}^*$ invariant under transformations replacing at most a finite number of coordinates $\omega_1, \omega_2, \cdots, \omega_k$ of a point of the set by others from $\Omega_1, \Omega_2, \cdots, \Omega_k$ respectively. $\mathfrak{F}^*$ is the family of all sets of $\mathfrak{B}^*$ which, for every positive integer k, can be expressed in the form:

$$\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{k-1} \times B_k^* \qquad (B_k^* \in \mathfrak{B}_k^*).$$

Clearly $\mathfrak{F}^*$ is an additive class of sets.

LEMMA 3. *If* $m_n(\Omega_n) = \mu_n(\Omega_n) = 1$ $(n=1, 2, \cdots)$, *and if there exists a positive integer* n_0 *such that* $m_n = \mu_n$ *for* $n \geqq n_0$, *then* $m^* = \mu^*$ *on* $\mathfrak{F}^*$.

For, then $m_{n_0}^* = \mu_{n_0}^*$; and since every set of $\mathfrak{F}^*$ is of the form $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{n_0-1} \times E_{n_0}^*$ for a set $E_{n_0}^* \in \mathfrak{B}_{n_0}^*$, this is an immediate consequence of Lemma 1.

Let a set B_n be chosen from each family $\mathfrak{B}_n$ $(n=1, 2, \cdots)$. Denote by $\mathfrak{B}_n \cdot B_n$ $(n=1, 2, \cdots)$ the additive class of all sets of $\mathfrak{B}_n$ contained in B_n. The following theorem assumes that Kakutani's conditions hold only on the subsets B_n of Ω_n, with a supplementary condition (7)′ on the B_n, and states that the infinite product measures are *equal* on $\mathfrak{F}^*$.

THEOREM 2. *If there exists a positive integer* n_0 *such that*

$$(1)' \qquad m_n \sim m_n' \qquad \text{on } \mathfrak{B}_n \cdot B_n,$$

$$(2)' \qquad m_n(\Omega_n) = m_n'(\Omega_n) = 1,$$

$$(3)' \qquad \prod \int_{B_n} (\phi_n(\omega))^{1/2} dm_n(\omega) > 0,$$

$$(7)' \qquad \prod m_n(B_n) > 0, \qquad \prod m_n'(B_n) > 0,$$

all for $n \geqq n_0$, *then* $m^* = m^{*\prime}$ *on* $\mathcal{F}^*$.

It is no restriction to suppose $n_0 = 1$. For if $n_0 > 1$, let μ_n and μ_n' be arbitrary measures satisfying (1)', (2)', and (7)' for $n < n_0$ (for example $\mu_n(E) = \mu_n'(E) = 1$ if E contains a fixed point $\omega_{n_1} \in B_n$, and $\mu_n(E) = \mu_n'(E) = 0$ if $\omega_{n_1} \notin E$), and $\mu_n(E) = m_n(E)$, $\mu_n'(E) = m_n'(E)$ for $E \in \mathcal{B}_n$ $(n \geqq n_0)$. Then assuming the theorem for $n_0 = 1$, $\mu^* = \mu^{*\prime}$ on $\mathcal{F}^*$. Hence by Lemma 3, $m^* = m^{*\prime}$ on $\mathcal{F}^*$.

PROOF. Apply Lemma 2 to the sequences of measure spaces[2] $\{B_n, \mathcal{B}_n \cdot B_n, m_n\}$ and $\{B_n, \mathcal{B}_n \cdot B_n, m_n'\}$ $(n = k, k+1, \cdots)$ for each positive integer k. One finds $m_k^* \sim m_k^{*\prime}$ on $\mathcal{B}_k^* \cdot B_k^*$, where $B_k^* = B_k \times B_{k+1} \times \cdots$. Since every set of $\mathcal{F}^*$ is of the form $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_{k-1} \times E_k^* f$ or some $E_k^* \in \mathcal{B}_k^*$, it follows from Lemma 1 that $m^* \sim m^{*\prime}$ on $\mathcal{F}^* \cdot A_k^*$, where

$$\begin{aligned} A_k^* &= \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{k-1} \times B_k \times B_{k+1} \times \cdots \\ &= \bigcap_{j=1}^{\infty} \Omega_1 \times \Omega_2 \times \cdots \times \Omega_{k-1} \times B_k \times B_{k+1} \times \cdots \\ &\qquad \times B_{k+j} \times \Omega_{k+j+1} \times \cdots . \end{aligned}$$

Let $A^* = \bigcup_{k=1}^{\infty} A_k^* = \lim_{k \to \infty} A_k^*$; and let E^* be a set of $\mathcal{F}^* \cdot A^*$ such that $m^*(E^*) = 0$. Then

$$\begin{aligned} E^* &= E^* \cdot A_1^* \cup E^* \cdot A_2^* \cup \cdots , \\ m^{*\prime}(E^*) &\leqq m^{*\prime}(E^* \cdot A_1^*) + m^{*\prime}(E^* \cdot A_2^*) + \cdots . \end{aligned}$$

But $m^{*\prime}(E^* \cdot A_k^*) = 0$ $(k = 1, 2, \cdots)$ since $m^* \sim m^{*\prime}$ on $\mathcal{F}^* \cdot A_k^*$. Hence $m^{*\prime}(E^*) = 0$. Similarly $m^*(E^*) = 0$ if $m^{*\prime}(E^*) = 0$, $E^* \in \mathcal{F}^* \cdot A^*$. Thus $m^* \sim m^{*\prime}$ on $\mathcal{F}^* \cdot A^*$. But the measure (m^* or $m^{*\prime}$) of any set of $\mathcal{F}^*$ is either 0 or 1 [4; 6, p. 60]. Also

$$m^*(A^*) = \lim_{k \to \infty} m^*(A_k^*) = \lim_{k \to \infty} \prod_{k}^{\infty} m_n(B_n) = 1,$$

and similarly $m^{*\prime}(A^*) = 1$. Therefore the measure of any set of $\mathcal{F}^* \cdot A^*$

[2] The triple $\{\Omega, \mathcal{B}, m\}$ denotes a measure space Ω, and an additive class of sets $\mathcal{B}$ on which a measure m is defined.

is either zero or one, so that $m^* = m^{*\prime}$ on $\mathcal{J}^* \cdot A^*$. Since also $m^*(E^*) = m^*(E^* \cdot A^*)$ and $m^{*\prime}(E^*) = m^{*\prime}(E^* \cdot A^*)$ if $E^* \in \mathcal{J}^*$, $m^* = m^{*\prime}$ on $\mathcal{J}^*$. This completes the proof of Theorem 2.

3. **An application.** Let $\{X_n\}$ and $\{Y_n\}$ be sequences of independent real-valued random variables with distribution functions V_n and W_n $(n = 1, 2, \cdots)$, and corresponding linear measures m_n and m_n'. Let the sets Ω_n of the preceding discussion here all be R_1, the set of real numbers. $\Omega^* = R_1 \times R_1 \times \cdots$ is the space of sequences $(\omega_1, \omega_2, \cdots)$, $-\infty < \omega_n < \infty$ $(n = 1, 2, \cdots)$. Each of the families $\mathcal{B}_n$ may be taken here as the family of Borel sets on $\Omega_n = R_1$. For $B_n \in \mathcal{B}_n$, $m_n(B_n)$ represents the probability of X_n belonging to B_n; for $B^* \in \mathcal{B}^*$, $m^*(B^*)$ represents Pr $\{(X_1, X_2, \cdots) \in B^*\}$.

Given a limit theorem whose conclusion is a property (P) of m^* on sets invariant under changes affecting at most a finite number of coordinates (that is, sets of $\mathcal{J}^*$), Theorem 2 gives sufficient conditions that $m^{*\prime}$ have the same property. That is, Theorem 2 gives conditions sufficient in order that $\{Y_n\}$ may satisfy the limit theorem if $\{X_n\}$ does, and in this way extends any known conditions sufficient for the limit theorem.

Let us refer to such a limit theorem as a limit theorem $(P; m^*, \mathcal{J}^*)$. Then as it applies to sequences of independent random variables, Theorem 2 becomes:

THEOREM 2′. *If there exist a positive integer n_0 and sets $B_n \in \mathcal{B}_n$ $(n \geqq n_0)$ such that*

(1)′ Pr $\{X_n \in E_n\} = 0$ *if and only if* Pr $\{Y_n \in E_n\} = 0$ *for* $E_n \in \mathcal{B}_n \cdot B_n$,

(3)′ $$\prod \int_{B_n} (\phi_n(\omega))^{1/2} dm_n(\omega) > 0,$$

(7)′ $$\prod \Pr\{X_n \in B_n\} > 0, \qquad \prod \Pr\{Y_n \in B_n\} > 0$$

(*or equivalently,* $\sum \Pr\{X_n \notin B_n\} < \infty$, $\sum \Pr\{Y_n \notin B_n\} < \infty$)

for all $n \geqq n_0$, then a necessary and sufficient condition that $\{Y_n\}$ obey a limit theorem $(P; m^{*\prime}, \mathcal{J}^*)$ *is that $\{X_n\}$ obey it.*

Many well known limit theorems are of this type. The strong law of large numbers, the law of the iterated logarithm, and Kolmogoroff's theorem on the probability of convergence of sequences of independent random variables are examples. The last named is a limit theorem $(P_1; m^*, \mathcal{J}^*)$, where (P_1) is the property:

(P_1) $\quad m^*(E_1^*) = 1, \qquad$ where $\quad E_1^* = \left\{ \sum_1^\infty X_n \text{ converges} \right\};$

that is, E_1^* is the set of points $(\omega_1, \omega_2, \cdots)$ of Ω^* such that $\sum_1^\infty \omega_n$ converges. Evidently $E_1^* \in \mathcal{F}^*$. The law of the iterated logarithm is a limit theorem $(P_2: m^*, \mathcal{F}^*)$, where (P_2) is the property:

(P_2) $\qquad m^*(E_2^*) = 1,$

$$\text{where} \quad E_2^* = \left\{ \limsup_{n\to\infty} \sum_1^n X_k / [s_n(2 \log\log s_n)^{1/2}] = 1 \right\}.$$

Here s_n denotes the sum of the second moments about the mean (the mean is here taken to be zero) of the first n random variables (cf. [5, p. 823]). Again $E_2^* \in \mathcal{F}^*$, if $s_n \to \infty$. The strong law of large numbers is a limit theorem $(P_3; m^*, \mathcal{F}^*)$, where P_3 is the property:

(P_3) $\quad m^*(E_3^*) = 1, \qquad$ where $\quad E_3 = \left\{ 1/n \sum_1^n X_k = 0 \right\}$

(here again the mean of X_k is assumed to be zero). Clearly $E_3^* \in \mathcal{F}^*$.

Theorem 2′ contains the following standard device. Let $\{Y_n\}$ be a sequence of independent random variables, and $\{B_n\}$ a sequence of sets $(B_n \subset \Omega_n, B_n \in \mathcal{B}_n)$ such that $\sum_1^\infty \Pr\{Y_n \notin B_n\} < \infty$. Define the sequence of independent random variables $\{X_n\}$ such that $X_n = Y_n$ on B_n, that is, so that for each positive integer n and for each $E \in \mathcal{B}_n \cdot B_n$, $m_n(E) = m_n'(E)$. Then, if $\{X_n\}$ obeys a limit theorem $(P; m^*, \mathcal{F}^*)$, so does the sequence $\{Y_n\}$. For then $\phi_n(\omega) = 1$ on B_n, and also $\Pr\{X_n \notin B_n\} = 1 - \Pr\{X_n \in B_n\} = 1 - \Pr\{Y_n \in B_n\} = \Pr\{Y_n \notin B_n\}$, hence the hypotheses of Theorem 2′ are satisfied.

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